

MOMENT MAPS AND SURJECTIVITY IN VARIOUS GEOMETRIES

organized by
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Workshop Summary

In the past thirty years, tremendous progress has been made in the study of moment maps, symplectic quotients, and the question of surjectivity of the Kirwan map. In recent years, similar questions have arisen in fields other than symplectic geometry: contact, hyper-Kähler, Sasakian, and 3-Sasakian geometry. While some headway has been made in understanding moment maps and surjectivity in these fields, there remain many open questions.

Our primary goal for this workshop was to explore phenomena that are well understood in symplectic geometry but are more puzzling in these new settings. Our secondary goal for the workshop was to foster collaborations across these fields. To achieve these, we brought together experts from these different fields within geometry to discuss the most recent results and to compile a list of open problems and conjectures. This group of mathematicians consisted almost exclusively of geometers. Within geometry, however, it is a diverse group of people who might not otherwise convene. Among the participants, we also included a number of graduate students, recent graduates and newcomers to the fields.

The structure of an ARCC workshop was very well-suited to our objectives and needs. At the AIM staff's advice, we arranged for a number of introductory lectures. This allowed people from different areas to get a more equal footing. Participants reviewed these lectures very highly in their feedback. In addition to these introductory lectures, we ran several "Ask the Expert" sessions. This provided an opportunity to ask very specific questions, to see examples worked out carefully, and to foster discussion. In the future, we would try to have smaller sessions so that more participants could be actively involved. In particular, the graduate students were particularly quiet during these sessions. We also had three moderated discussion sessions where open questions and problems were posed and discussed. This type of session was new for most participants and it took them some time to adjust to the format. However, participants almost universally agreed that these sessions were highly productive. Finally, we allowed for several large blocks of time for small group meetings. During this time, some small groups started to work on some of the questions raised during the discussion sessions, while others used the time to continue already existing collaborations. In general, the fairly open schedule encouraged many useful discussions and paved the way for new projects and collaborations.

The conference opened with an introductory lecture on the history of moment maps and surjectivity in symplectic geometry. Several more lectures on symplectic geometry followed throughout the week. In addition, during the discussion sessions, three symplectic questions came to the fore, and were discussed during the small group meetings. The first such question was whether surjectivity can be proved using Karshon's abstract moment maps. There is a flawed proof in the book of Ginzburg, Guillemin and Karshon. The small group made some

progress in finding a different proof. This also led us to wonder if abstract moment maps can be defined and used in the other geometries. The second question was how to use surjectivity techniques to understand the Chen-Ruan orbifold cohomology of symplectic reductions. This has been answered for torus reductions, and during the small group session, some progress was made in the case of non-abelian reductions. The third question was to what extent Kirwan surjectivity, and corresponding kernel computations, hold with \mathbb{Z} coefficients. In all cases, and for several other problems, we have written down explicitly several conjectures and open problems.

On the hyper-Kähler front, we had four lectures, starting at a reasonably elementary level and finishing with a long list of open problems and conjectures. In this area, it is most likely that techniques from symplectic geometry should have some bearing. There were at least two points of progress in this field during the week. First, one participant arrived at the conference with a twenty year old manuscript of Frances Kirwan that gave two proofs of hyper-Kähler surjectivity under some hypotheses on a gradient flow. Kirwan had found a mistake in the first proof, and so never published the work. Some participants had been aware of the manuscript, and of the purported mistake, but this was a surprise to most. During the small group meetings, one group carefully read through the supposedly correct proof to see if it is valid. Although the hypotheses are rather restrictive, the result should at least apply to a reduction of $T^*\mathbb{C}^n$ by the linear action of a compact Lie group. This is already a hyper-Kähler reduction where surjectivity is not known to hold. This group plans to continue checking Kirwan's proof. The second point of progress involved the understanding of integration in the non-compact hyper-Kähler setting. There was one lecture on this subject, and several small group discussions ensued. Some experts from symplectic geometry had ideas about how to improve and better explain the results on the hyper-Kähler side.

Finally, in contact, Sasakian and 3-Sasakian geometry, we had an introductory lecture and two more advanced lectures. In this geometry, surjectivity does not hold in general: there are counter-examples in contact geometry. To understand the topology of quotients, we will likely need to employ new techniques. In the symplectic category, surjectivity results are proved using the norm-square of the moment map as a kind of Morse function on the manifold at hand. For contact moment maps, however, the norm-square of the moment map cannot be used in such a way. One can still ask for partial results, and the expert in 3-Sasakian geometry made a conjecture for the strongest result that he expects to be true. During a small group session, he and an expert in hyper-Kähler geometry made significant progress in this direction.

The progress made at this ARCC workshop has also helped the organizers prepare for an upcoming conference. Holm and Lerman, along with three Canadian colleagues, are organizing a subsequent conference at the Banff International Research Station in May 2005. We have used our time at AIM in part as a lead-in to that conference. We hope that significant progress will be made on the open problems and conjectures discussed at AIM, and will be presented in Banff. This work will directly aid us in Banff, where we will study natural metrics on such quotients. We will then use these results to search for explicit descriptions of special metrics on Kähler and Sasakian manifolds. In addition, we have found many of the unorthodox sessions and ideas at AIM to be very useful for fostering discussion and collaboration. We intend to employ these innovative methods at the Banff meeting.

In conclusion, the ARCC workshop on moment maps and surjectivity in various geometries was a definite success. In their feedback, all participants reported finding new problems to work on and receiving good advice on projects currently underway. Everyone participated in at least one small group session, and we expect many of these sessions to lead to longer-term collaborations and substantial progress in the field. We owe a large debt of gratitude to the AIM staff for organizing and facilitating the workshop and to AIM and the NSF for funding the conference.