Sharp Thresholds for Mixing Times

organized by
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Workshop Summary

Summary of mathematical progress at "Sharp Thresholds for Mixing Times" workshop

The tightly focused nature of the AIM workshop "Sharp Thresholds for Mixing Times" resulted in mathematical progress in at least three different areas. The goal of the workshop was to discuss the question of when is there a sharp threshold for the mixing time for a Markov chain. In particular, one of the organizers (Y. Peres) has recently conjectured that a necessary and sufficient condition for the L_1 mixing time T_{mix} to have a sharp threshold is that the relaxation time T_{rel} (also known as the inverse spectral gap) be such that $T_{\text{rel}} = o(T_{\text{mix}})$.

One area of progress that was made during the workshop was on clarifying the exact nature of Peres' conjecture and discussing related results and examples. Progress towards this began in preparations for the workshop. Saloff-Coste had previously shown that the analogous conjecture holds in L_p for $1 < p < \infty$ (and in the case of reversible chains, also holds for L_∞). The nature of the workshop caused him to refine some of these results, which he presented in one of his talks. They came as something of a surprise even to some participants who have been working in the area for quite some time. For example, in an earlier talk, Revelle gave an example of a Markov chain for which the L_∞ mixing time had a sharp threshold even though the location of the threshold was unknown, and while Saloff-Coste's result gives many examples of this type, the special case mentioned by Revelle still surprised even some of the senior members of the audience. One reasons why these results were surprising is that the precise shape of the cutoff was computed in the early examples of Markov chains with sharp thresholds, while Saloff-Coste's result gives a way of proving the existence of a threshold even without enough information to be able to explicitly compute its shape.

An important step towards clarifying Peres' conjecture came from an unplanned source: Aldous had stated before the workshop that he preferred not to give a talk, but after hearing some of the talks on the opening day, he changed his mind. The flexible schedule of the workshop allowed us to accommodate his request, and he then presented an example showing that in the case of L_1 mixing time on a non-vertex transitive graph Peres' conjecture has to be modified, replacing the worst case starting points with an appropriate averaging on the starting point. In this context Aldous also presented an earlier conjecture of his about not just whether or not there is a sharp threshold, but also what the shape of the threshold should be (a conjecture that was new to most workshop participants).

A second area of success of the workshop involved bringing together researchers who have been studying mixing times for many years, such as Diaconis, Aldous, Fill, and Saloff-Coste, with researchers newer to the field who have recently made key contributions, such as Wilson, Randall, and Morris. These two groups of participants have different perspectives on the subject, and their interactions resulted in finding new connections between different ideas. One such link that was found during the workshop is between the concept of evolving sets, which Morris used in his work on bounding the mixing time of the Thorp shuffle, and the notion of strong stationary duality, introduced more than a decade earlier by Diaconis and Fill. The flexible nature of the schedule helped this development, as Fill's talk was not initially planned and was added in response to some comments on earlier talks. As Fill prepared his talk after hearing Morris's talk, while thinking about the ideas presented in Morris' talk, he was able to discover this link, which is yet to be fully elucidated and exploited.

A third important development at the workshop came from Wilson's talk about a new technique for finding lower bounds for mixing times. Most of the existing theory focuses on obtaining upper bounds for mixing times, with emphasis on reversible chains; lower bounds were often obtained in a more ad-hoc manner. Thus two important features of Wilson's talk were the systematic approach to lower bounds for mixing times and the emphasis on non-reversible chains. One application of Wilson's methodology (which actually required refining it) was presented in a short talk by D. Randall.

Although a number of participants were already familiar with Wilson's methods, he presented a number of examples in which the upper and lower bounds for the mixing time differ by a constant. In all these examples a sharp threshold is conjectured but not yet proved so they provide concrete challenges for the future.