# Mapping theory in metric spaces

organized by
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Workshop Summary

Recent years have witnessed remarkable advances on basic questions concerning uniqueness, extendibility, embeddability, uniformization and extremality of mappings in a variety of regularity classes. A persistent source of complexity is the influence of the geometry of the target space: problems whose solution is straightforward in the scalar-valued case become more intricate for vector-valued targets, even more so in case the target is a (nonlinear) manifold or metric space. The purpose of the workshop *Mapping Theory in Metric Spaces*, held at AIM January 9-13, 2012, was to bring together researchers with differing backgrounds and expertise to highlight common techniques and methods and leverage existing knowledge towards the successful resolution of interdisciplinary problems related to Sobolev mappings on and between metric spaces, Lipschitz and optimal Lipschitz extensions, and bi-Lipschitz and quasisymmetric embeddings and uniformization.

The mornings of the workshop were mostly dedicated to 'tutorial' lectures given by some of the senior participants of the workshop to address the main themes in a way that was accessible to all participants. The purpose of these talks was to provide the audience with a self-contained survey of the main topics while at the same time creating momentum for the ensuing discussions and problem solving sessions. Tutorial lectures given by Urs Lang and Charlie Smart focused on Lipschitz extension problems and optimal Lipschitz extension problems, respectively. Lang discussed, in particular, classical and contemporary extension problems for Lipschitz mappings, and explained the effects of the geometry of the target space. Lang's lecture also briefly touched on connections to geometric functional analysis. Smart discussed absolutely minimizing Lipschitz extensions (AMLE) and tight extensions, and explained the connections to PDEs and the calculus of variations. Mario Bonk and Guy David gave tutorial lectures related to uniformization and analysis in metric spaces and, respectively, parameterizations and rectifiability. Bonk gave an introduction to uniformization theorems, from the classical theory to recent results on quasisymmetric uniformization of metric spheres. He highlighted the use of circle packings and explained connections to geometric group theory and, in particular, to Cannon's conjecture. David lectured on Reifenberg's theorem and recent variants and generalizations and discussed analogies to the Traveling Salesman problem. Tutorial lectures on Sobolev mappings as well as on quasiconformal mappings and mappings of finite distortion were held by Piotr Hajlasz and Pekka Koskela. Hajlasz gave an introduction to Sobolev mappings between manifolds and, more generally, between metric spaces and, in particular, recent results and questions concerning the density of Lipschitz maps in the space of Sobolev maps. Koskela gave an extensive introduction to quasiconformal maps and mappings of finite distortion. Thierry de Pauw gave a lecture on Whitney's integration theory, in particular, on flat chains and co-chains. These have recently been used in the context of biLipschitz parameterization questions by Heinonen and Sullivan

and by Heinonen and Keith. Finally, Urs Lang gave an introduction to injective metric spaces and injective hulls. These have found applications in reconstruction of phylogenetic trees.

The afternoons of the workshop were almost entirely devoted to discussions, mostly in smaller groups. On Monday afternoon, in a long and very active session, open problems and questions were solicited from the participants. On the afternoons of Tuesday - Thursday smaller groups of 3 - 8 participants worked on and discussed the open problems gathered on Monday. Finally, on Friday afternoon, the list of problems created on Monday afternoon was extended and amended in another "problem session". The list of problems which was compiled will appear on AIM's webpage. Some of the topics and problems discussed in the afternoon sessions are described in more detail below.

**Extension problems for Lipschitz mappings:** Prompted by a question raised in Lang's lectures, a small group discussed during two afternoons whether the pair $(L^2, L^1)$ has the Lipschitz extension property. A second group focused on questions concerning the dependence on parameters for Lipschitz extensions and optimal Lipschitz extensions. Charlie Smart gave a brief presentation of an argument by Kopecka, where a continuous dependence is established. Prompted by Lang's lectures on Lipschitz extensions, questions about the extendibility of bi-Lipschitz mappings were also raised during the workshop. In particular, one group investigated Daneri–Pratelli's proof of the fact that every $L$-bi-Lipschitz embedding of $S^1$ into $R^2$ extends to a $CL^4$-bi-Lipschitz map of $R^2$ and investigated possible improvements.

**Embedding problems:** One group investigated a possible deterministic proof of Naor–Neiman's probabilistic proof of a strengthened Assouad-type embedding theorem for metric spaces. Various ideas were proposed for a deterministic proof. A second group discussed the role of differentiation in proving bi-Lipschitz non-embeddability, especially of snowflake metric spaces. It was discussed whether the snowflake-ed Heisenberg group admits a biLipschitz embedding into a low-dimension Euclidean space. The discussion in this group also led to a connection between biLipschitz embeddability and the existence of Ahlfors $m$-regular topological submanifolds of dimension $m$ in Euclidean space.

**Uniformization and parameterization:** During the workshop, approaches to local and global bi-Lipschitz and bi-Hölder Euclidean parameterizations were discussed at length. In particular, approaches to biLipschitz parameterizability using tools from geometric measure theory (flat forms and Cartan–Whitney presentations) as well as from conformal geometry ($Q$-curvature) were explored.

**Rigidity for $p$-harmonic functions and mappings:** Prompted by a question by Mario Bonk, a group worked during two afternoons on questions related to $p$-harmonic and $\infty$-harmonic functions and mappings. In particular, the aim of the group was to find proofs of Liouville type theorems (characterizing entire bounded (and linear growth) $p-$harmonic functions) sufficiently simple so that they could be generalized to certain discrete settings. As was emphasized, an elementary proof might allow an extension to (certain classes of) discrete groups, which might in turn have applications to problems in geometric group theory. The group was indeed able to find such an elementary proof (in the case of bounded $p-$harmonic functions) which might be applicable to a rather wide class of spaces. The group further discussed how such Liouville type theorems would play a role in a new proof of the rigidity of 1-quasiconformal mappings.

**Quantitative aspects of regularity vs. geometry of spaces of mappings:** One group focused during two afternoon sessions on optimal regularity classes for the solutions of certain rigidity problems. PDE-based approaches to rigidity were also discussed.

In conclusion, the workshop has brought together experts from different fields in an attempt to initiate interaction and exchange between researchers who otherwise would have little opportunity to meet. We believe that this attempt was successful and that discussions begun at the workshop will continue and lead to new collaborative work at the intersection of various fields. The workshop has received very positive feedback from the participants and the creation of a list of open problem was met with enthusiasm.