

LOW EIGENVALUES OF LAPLACE AND SCHRÖDINGER OPERATORS

organized by

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Workshop Summary

The workshop was designed to focus on two main problem areas, both involving the eigenvalues of Laplace and Schrödinger operators. Specifically, investigation centered on

- (1) Sharp constants in the classical Lieb-Thirring inequalities, and
- (2) Optimal lower bounds for the gap between the two lowest eigenvalues of Laplace and Schrödinger operators, specifically the conjectured optimal lower bound $3\pi^2/d^2$ for a bounded convex domain of diameter d in n dimensions (with the potential being convex on the domain, in the case of a Schrödinger operator).

Roughly speaking, and as far as the formal aspects of the workshop went, we began with an emphasis on Lieb-Thirring inequalities for the first two days and then brought in questions and approaches to the optimal eigenvalue gap problem beginning on the third day. But threads that began in small group discussions and working seminars generally continued throughout the week. Thus, in the latter stages of the week we still had vigorous discussion groups going on Lieb-Thirring matters as well as the more newly developing discussion groups centering on approaches to gap inequalities.

CHRONOLOGY

To be more specific, the workshop began on Monday, May 22 with a very well-presented (and also well-received) general survey talk by Michael Loss on Lieb-Thirring inequalities. Still before lunch the entire group had a broad-ranging discussion of Lieb-Thirring inequalities, including mention of a variety of open problems. The discussion was led by Eric Carlen and set a good tone for the workshop, especially in the sense of encouraging informal interactions and the free exchange of ideas. This was followed in the afternoon by a talk by Timo Weidl that took us into the midst of the problems and ideas concerning sharp constants for these problems. Later we broke into two smaller discussion groups, which focused on two intriguing problems that had arisen in the earlier discussions. One of these was the “Ovals Problem” and the other “Lieb-Thirring with remainder terms”.

On the second day of the workshop Rupert Frank led off with an introduction to his recent work with Tomas Ekholm (also a workshop participant) on Lieb-Thirring-Hardy inequalities. This was in a sense breaking news and took us “up to the minute” on the status of current work on Lieb-Thirring inequalities. This talk was followed by another full-group discussion session, this time led by Evans Harrell. This discussion centered especially on the best currently known constants for Lieb-Thirring inequalities and on conjectures for what should be the sharp constants. By the end of this discussion we had a table on the board

(which we then recorded) that summarized our state of knowledge on best constants for the various cases of the Lieb-Thirring inequality (covering changes of dimension, exponent, best constants to date, whether or not they are known to be sharp, and what the sharp constants are conjectured to be in the cases that remain open). After lunch we had two working groups, one on the Ovals Problem and the other on better constants for Lieb-Thirring inequalities. Later we all got back together and had a general discussion, which included some ideas of Carlo Morpurgo on improving the constant in the Cwikel-Lieb-Rosenbljum bound (which can be viewed as the special case of a Lieb-Thirring inequality where the exponent γ has been set equal to 0). We also had a very nice brief presentation from Elliott Lieb on his recent work with Frank, Laptev, and Seiringer on Lieb-Thirring inequalities for complex potentials (and hence for non-selfadjoint Schrödinger operators).

The third day began with a general presentation from Rodrigo Bañuelos, supported by brief presentations from Pedro Mendez-Hernandez and Robert Smits, on the gap inequality, and in particular on their approach to the problem via heat kernels and multiple integrals. A discussion of gap problems then occurred, led by Mark Ashbaugh. Participants threw out many conjectures and ideas for attacking them, which were duly recorded. In the afternoon we broke up for excursions, or just had some free time.

On the fourth day we began with a talk by Antoine Henrot on other approaches to gap and eigenvalue optimization problems, including analytical approaches and the technique of shape optimization. Following that, we broke into smaller groups, this time with the main groups focusing on the Ovals Problem and the other focusing on the P-function approach to optimization problems (including a nice presentation by Gerard Philippin on the general approach to finding and using P-functions). That afternoon Rick Lavine gave a talk on his proof of the gap conjecture for the one-dimensional Schrödinger operator, which is the only case of the gap conjecture that's actually been proved. We ended the day with further discussions, both in the full group and in smaller groups.

On the final day of the workshop Lennie Friedlander gave a talk on his fundamental result comparing Dirichlet and Neumann eigenvalues of the Laplacian. Also we had several discussions in smaller groups, with one promising discussion centering on the possibility of doing analogs of the Lieb-Thirring theory for the linear spectral problem associated to the nonlinear Benjamin-Ono equation, the possibility for which had been raised by Rafael Benguria. We continued throughout the day with various small group discussions and finally the whole group got together near the end of the day to try to summarize the progress we'd made on the various problems raised and where our discussions had led.

CONCRETE OUTCOMES

There have already been several interesting and promising ideas that have come out of the workshop. We expect more to develop or come to light over time, as participants are able to reflect on what they have learned and as they pursue the discussions and collaborations that were begun at the workshop. For now we list the following three items:

- (1) The discussion of the Ovals Problem generated a lively and fruitful discussion throughout the week, with many people contributing thoughts or information on one aspect or another of the problem. The fact that the ovals problem admits a degenerate class

of minimizers consisting of the circle as well as ellipses led to the following promising idea: that one should somehow employ a “curve-shortening approach” to solving the problem, but within the context of *Affine Geometry*. In Affine Geometry there is an Affine Isoperimetric Inequality which has precisely as its optimizers the class of all ellipses (including circles). The idea would be to use “Affine Curvature Flow” to make progress on the Ovals Problem. This engendered a lively discussion and presentations to that subgroup that included Rafe Mazzeo, Eric Carlen, Larry Thomas, Rafael Benguria, Almut Burchard, Helmut Linde, and many others, and will likely lead to some developments along these lines. Specifically, we expect to prove that the lowest eigenvalue of the Schrödinger operator on a closed, smooth curve, of fixed length, with periodic boundary conditions and with a potential given by the curvature squared is monotonically decreasing under the affine curvature flow, and in this manner, we hope for a proof of the isoperimetric inequality for ovals. Special attention will have to be paid to preserving the length of the curve under the flow.

- (2) Helmut Linde, a very promising young researcher who had just completed his Ph.D. under Rafael Benguria in Chile, got to see and interact first-hand with many of the world leaders in his field as a participant in the workshop. During the workshop he was trying to decide between taking up a postdoctoral position or pursuing a career in business. The excitement and stimulation provided by the workshop was enough to convince Helmut to stay in academics.
- (3) During the latter part of the workshop, in further discussions of Lieb-Thirring inequalities, Rafael Benguria suggested that there should be an entire “Lieb-Thirring theory” associated with the Benjamin-Ono equation. That equation is a nonlinear integro-differential equation, but it has associated with it a linear spectral problem (a linear system) with a “potential”, much like the Korteweg-de Vries equation has associated with it a linear spectral problem for the one-dimensional Schrödinger operator. This generated considerable interest, and Rafael was prevailed upon to present some brief background on the Benjamin-Ono equation and some further explication of his ideas. By the end of the week Tomas Ekholm, Rupert Frank, and Dirk Hundertmark had already made substantial progress on the problem. In particular, they believed they had obtained the analog of the Aizenman-Lieb “monotonicity toward best constants” result from the standard Lieb-Thirring theory.

We also expect that our AIM website giving information on the workshop and on the various open problems that arose will continue to be helpful to the workshop participants and to the general mathematical community for the foreseeable future.

CONCLUDING REMARKS

Finally, we would like to state that we organizers all enjoyed the distinctive approach that AIM brings to its workshops and that we very much appreciated the guidance and assistance of the entire AIM staff. As the week went along, we came to see firsthand the wisdom of their methodology. We believe we speak for all the participants in saying that it was an unusually productive week, most notably in terms of knowledge shared and ideas generated.