Lipschitz metric on Teichmueller space

organized by
Fanny Kassel, Athanase Papadopoulos, Kasra Rafi, and Jing Tao

Workshop Summary

The purpose of the AIM workshop "Lipschitz metric on Teichmüller space" was to highlight some recent developments and to embark on some new directions in the study of Teichmüller space from the point of view of the Lipschitz metric. The workshop gathered some specialists from several fields of study, such as Teichmüller theory, Finsler geometry, Lie groups, Outer space, as well as newcomers to the theory, including young researchers and graduate students.

The first order of business on the first day was to dedicate the workshop to the late William Thurston, who first introduced the Lipschitz metric on Teichmüller space in an unpublished manuscript, "*Minimal stretch maps between hyperbolic surfaces*," in 1986. The workshop participants then came to a consensus that this metric should be renamed the "Thurston metric", to be denoted by d_{Th}. This was appropriate not only because Thurston was responsible for the genesis of the field, but also because this was in keeping with the naming tradition of the other metrics on Teichmüller space, such as the Teichmüller metric or the Weil–Petersson metric.

The first morning proceeded with two introductory talks on the Thurston metric by two of the organizers. The first talk, presented by Athanase Papadopoulos, focused on some fundamental definitions and concepts, such as the definition of the Thurston metric, Thurston's characterization of the metric in terms of length ratios of curves, maximally stretched laminations, stretch maps, some aspects of asymmetry, and symmetrization. This was followed by a talk by Kasra Rafi on comparisons and contrasts between the Thurston metric and the Teichmüller metric. These two talks were intended to familiarize the non-experts with the basic objects as well as set up some groundwork for later discussions.

The first afternoon session was devoted to questions and answers. The questions were solicited from the participants throughout the morning. It turned out that, although some of the questions were about basic concepts and appropriate experts were able to answer them, many of the questions were in fact open questions. At the end of the afternoon, the organizers put up all the open questions on the board, then an election process was initiated to assess the interests of the participants. Since there were so many questions and since it was clear that everyone was interested in several of the questions, the election process was relegated to the happy hour, so that people could discuss freely while deciding on their favorite questions (more than one favorite was allowed). At the end of the day, the organizers recorded the questions (about 10) that generated the most interests. These questions naturally became a proposed problem list that helped shape the discussion sessions for the rest of the week.

For the rest of the week the workshop kept to a schedule of two research talks in the morning (each talk running about one hour and fifteen minutes) and then afternoon discussion sessions on open problems that generally occurred on the level of about four

to five groups with about three to seven participants each. To decide on the groups, the organizers would put up a list of proposed problems on the board and solicit new ones from the participants. Then one round of voting would focus the topics of the discussions to the four or five most popular, and another round of voting would divide the participants into the topics of their choice. At the end of the day, each group would choose a representative to report on their activities the next day at the beginning of the afternoon session.

This workshop was definitely successful in igniting exchanges between the experts from the different areas and broadening the interest in the field. The verbal feedbacks we received from the participants were very positive. It appears that the format of the AIM workshop was particularly fitting for this topic. Some of the discussions begun here have continued and may evolve into collaborative work. At the end of the workshop, there were unsurprisingly more questions than answers. Some of the problems were decidedly very hard; others appeared more tractable. It is clear that the work on these problems that was made during the week opened new perspectives for future results. In the last afternoon, a list of interesting open problems was compiled. One of the participants, Weixu Su, was chosen to summarize the discussion and write down the problems more formally. These problems will be sent to the AIM staff so that it can appear on the AIM website, as a service to the mathematical community.

Schedule

Monday.

(1) **Athanase Papadopoulos**: Introduction to the Thurston metric (definitions, symmetrizations, stretch maps, maximally stretched laminations, reverse Thurston geodesics)
(2) **Kasra Rafi**: Comparison between the Thurston metric and the Teichmüller metric (hyperbolic length v. extremal length, Minsky's product region theorem, bounded projections, etc.)
(3) **Question-and-answer session**
(4) **A list of proposed problems**
    (a) Comparison between the Thurston and the Teichmüller metric
    (b) Detailed study of the punctured torus case
    (c) Envelopes of geodesics in the Thurston metric
    (d) Inductive behavior of Thurston geodesics
    (e) Algorithms to find the maximally stretched lamination
    (f) Infinitesimal properties of the Thurston metric
    (g) The study of the horofunction boundary of the reverse metric
    (h) The Thurston metric for strata of singular flat metrics space
    (i) Masur's criterion for Thurston geodesic rays
    (j) Symmetrization of the Thurston metric
    (k) Geometry of Outer space

Tuesday.

(1) **Fanny Kassel**: Generalizations of the Thurston metric to a geometrically finite setting and to higher dimension

(2) **François Guéritaud**: Infinitesimal aspects of the Thurston metric (the Thurston sphere in the cotangent space as described in Thurston's paper, generalization to a convex cocompact setting, and applications to Lorentzian geometry)
(3) **Discussions in small groups**:
- Thurston geodesics in the punctured torus case / induction
- Algorithms to find the maximally stretched lamination
- Does the Thurston sphere in the cotangent space at $x \in \mathrm{Teich}(S)$ determine x?
- The backwards horofunction boundary of Teichmüller space for the Thurston metric
- The Thurston metric for flat surfaces

Wednesday.

(1) **Mladen Bestvina**: The Lipschitz metric on Outer space (the tension/Green graph and its train track structure, folding paths, standard geodesics, applications to the hyperbolicity of the complex of free factors and to subfactor projection)
(2) **Weixu Su**: The Finsler structures of the Thurston and Teichmüller metrics (description of the Finsler norms, stretch lines and Teichmüller lines, extension to surfaces of finite type with boundary)
(3) **Discussions in small groups**:
- Thurston geodesics in the punctured torus case
- Algorithms to find the maximally stretched lamination
- Dynamics: existence of Thurston geodesics that are dense in moduli space? How likely is it that a stretch path be recurrent?
- The backwards horofunction boundary of Teichmüller space for the Thurston metric
- Teichmüller spaces and the Thurston metric for surfaces of infinite genus
- Outer space: backwards folding lines, Masur's criterion for Outer space?

Thursday.

(1) **Dragomir Saric**: Surfaces of infinite type (the quasiconformal Teichmüller space, extremality, Strebel points, the Thurston bordification; the length spectrum Teichmüller space and its symmetrized Thurston metric; comparison between these two Teichmüller spaces)
(2) **Cormac Walsh**: The horofunction boundary of Teichmüller space (definitions and examples, the horofunction boundary of Teichmüller space for the Thurston metric, application to isometries, the horofunction boundary of Teichmüller space for the Teichmüller metric)
(3) **Discussions in small groups**:
    - Inductive behavior of Thurston geodesics/ introduction to Rafi's inductive description for Teichmüller geodesics
    - Algorithms to find the maximally stretched lamination
    - Dynamics: existence of Thurston geodesics that are dense in moduli space? How likely is it that a stretch path be recurrent?
    - The backwards horofunction boundary of Teichmüller space for the Thurston metric
    - Teichmüller spaces and the Thurston metric for surfaces of infinite genus
    - Outer space: backwards folding lines, Masur's criterion for Outer space?

Friday.

(1) **Misha Kapovich**: Vector-valued metrics on higher-rank Riemannian symmetric spaces (Cartan projections, conditions for triangular inequality)
(2) **Ania Lenzhen**: Coarse geometry of the Thurston metric (shadows of Lipschitz geodesics in the curve complex, Lipschitz and Teichmüller geodesics)
(3) **Open problems session** Everyone gathered together to give a final report for the small group discussions and to propose a list of open problems to be made available on the AIM website. The person in charge of taking notes and compiling the list of problems was Weixu Su.