The Kardar-Parisi-Zhang equation and universality class

organized by
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Workshop Summary

This workshop, sponsored by AIM and NSF was devoted to the study of the Kardar-Parisi-Zhang non-linear stochastic partial differential equation, and its universality class. Brownian motion is a continuum scaling limit for a wide class of random processes, and there has been great success in developing a theory for its continuum properties (such as distribution functions) and expanding the breadth of its universality class. More recently, a new universality class has emerged to describe a host of important physical and probabilistic models (such as one dimensional interface growth processes, interacting particle systems and polymers in random environments) which display unusual scalings and new statistics. This class is called the Kardar-Parisi-Zhang (KPZ) universality class and underlying it is, again, a continuum object – now a non-linear stochastic partial differential equation – which is known as the KPZ equation.

The purpose of this workshop was to build on very recent successes in understanding the KPZ equation and its universality class. There were two main focuses: Studying the integrability properties and statistics of the KPZ equation; and extending the universality of the KPZ equation. Talks and working groups were split evenly between these two focuses, with a great deal of interaction and cross-fertilization. The combination of great timing and a well balanced and energetic group resulted in a very productive and educational workshop. Two features which were particularly appreciated by participants were the "secret problems" session and the "big picture" session. The first session occurred on Monday when the organizers collected questions on paper scraps in which participants could ask fundamental questions they might be embarrassed to ask otherwise. The organizers read them over and, with the help of speakers, went through and answered each question. In the big picture session, the organizers led a brainstorming session in which the big open questions for the field were identified and explained to all participants. This occurred Thursday and helped put things in perspective and give people an outlook.

1.1 Studying the integrability properties and statistics of the KPZ equation Surprisingly, it is possible to give exact formulas for certain statistics associated with this non-linear stochastic PDE – such as its one-point distribution. The goal here was to understand the extend to which one can perform exact calculations using various techniques including the "Macdonald process" approach of Borodin-Corwin (announced at the workshop), the "tropical RSK correspondence" approach of O'Connell and Corwin-O'Connell-Seppäläinen-Zygouras, the ASEP Bethe ansatz approach of Tracy-Widom, and the slightly older replica approach.

Alexei Borodin kicked off the workshop by giving the first of two talks (the first on Monday and second on Wednesday) on “Macdonald processes” – a joint work with Corwin. Macdonald functions are symmetric functions which simultaneously diagonalize a large number of commuting operators. Due to certain non-negativity properties of Macdonald functions it is possible to specify measures on partitions and Young tableaux in terms of the functions. Certain observables of these measures can be computed using the commuting operators and when $t \to 0$, these can be combined into Fredholm determinant formulas for certain q-Laplace transforms.

Dynamics can be introduced on these Young tableaux so as to preserve the class of Macdonald process measures (though the specializations change in time). A projection of these dynamics is a Markov process called q-TASEP which received a great deal of attention during the workshop. In q-TASEP particles labeled x_i for $i \geq 1$ start at $x_i(t = 0) = -i$ and evolve according to a Markov process in which each particle attempts to jump one step right with rate $a_i(1-q^d)$ where d is the number of holes between the particle and the next particle to the right. One example of a q-Laplace transform from the Macdonald process work is that of $x_N(t)$. Just as Bertini-Giacomin showed ASEP with weak asymmetry scales to the KPZ equation, one of the group projects was to show that q-TASEP with $q \to 1$ converges in an appropriate scaling limit to the KPZ equation. This project met with success.

This limit is expected since, under certain specializations, these Macdonald processes degenerate as $q \to 1$ to the “Whittaker processes” introduced by O’Connell and Corwin-O’Connell-Seppäläinen-Zygouras in the study of the semi-discrete and discrete log-gamma directed polymers. The connection between these measures and polymers is a result of the “tropical Robinson-Schensted-Knuth correspondence” which was the main topic of Neil O’Connell’s Wednesday talk (he also provided a great deal of background on the RSK correspondence in its many guises). The Fredholm determinants from above now characterize the Laplace transforms of partition functions.

Expanding on asymptotics in Borodin-Corwin, one group studied the effect of having a few of the a_i’s (actually their limiting analogs) larger than 1 and all other equal to 1. In the zero-temperature polymer setting this type of perturbation leads to the Baik-Ben Arous-Péché (BBP) transition from random matrix theory. This was also found to arise in the study of positive temperature polymers. As Tom Alberts explained in his Tuesday talk, Alberts-Khanin-Quastel show that scaling the inverse temperature of a polymer model to zero as the system size grows leads to a universal limiting polymer model – the continuum directed random polymer whose partition function solves the stochastic heat equation and free energy solves the KPZ equation. Under this scaling (called “intermediate disorder” scaling) the same working group computed analogous formulas to those of the BBP transition but for the KPZ equation thus going from two known statistics for the KPZ equation to an infinite number.

The formulas of Borodin-Corwin provide an ansatz for solving certain quantum many body problems and the whole approach can be thought of as making rigorous mathematics out of the replica trick approach through which physicists compute moments of the partition function of a polymer and then try to recover the distribution (despite the fact that moments grow too fast). On Thursday Tomohiro Sasamoto explained another approach to making this replica trick rigorous, which was based off of a duality for ASEP with a many body system (a work in progress with Borodin and Corwin).

Timo Seppäläinen spoke on Friday about a related form of solvability for directed polymers which is a generalization of the classical “output theorem” (sometimes called Burkes theorem) of queuing theory. This provides an approach to prove fluctuation exponents for the free energy of solvable polymers for which such generalization hold – such as the semi-discrete or log-gamma discrete polymer. One group made a concerted effort to generalize this form of solvability to non-lattice polymers involving a weighted Poisson point potential.

Alan Hammond and Daniel Remenik both spoke on Thursday about the Airy line ensemble and the continuum statistics describing the path of the top line – the Airy_2 process.

1.2 Extending the universality of the KPZ equation Though it is recognized within physics literature as universal, the KPZ equation has only been shown to rigorously occur as the scaling limit of a few special models, for example, weakly asymmetric simple exclusion processes, and weakly rescaled polymers. In large part this is due to the fact that a well-posedness theory was lacking for the KPZ equation and thus it was only defined through the Hopf-Cole transform of the well-posed multiplicative stochastic heat equation. This meant that one was restricted essentially to models where a microscopic version of the Hopf-Cole transformation holds exactly. A good well-posedness theory for KPZ itself would provide a route to proving scaling limits to the KPZ equation for a wider class of models. In fact, Martin Hairer had announced an approach to such a theory just a few weeks prior to the workshop, and we were very lucky that he could give the participants an early introduction to his method. The idea in a nutshell is to write the KPZ equation as a finite number of complicated, but explicit processes, plus a stochastic equation which can be made sense of using rough path theory. The unpleasant "infinities" that need to be subtracted from KPZ all occur within the very explicit processes. So one has a very clear handle on them. Hairer gave two talks as a short minicourse introduction to the method. On Tuesday afternoon a group of six started working to see how the method might be applied to show that microscopic models converge to KPZ in the weakly asymmetric limit. As a first test case, we chose a class of semi-discrete versions of KPZ which are systems of coupled stochastic differential equations on the integer lattice. One particular model has a explicit product invariant measure. There was an open question as to how much the proofs would depend on the invariant measure. It was somehow expected that we would be able to show that this one converges to KPZ, but that there would be problems with the others which did not have explicit invariant measures. By the end of the week, it became apparent that the explicit invariant measure was not being used in the proofs (though this will require further confirmation). So it seems that the result that will come out of this is that one only needs a few basic symmetries on the discrete version of the nonlinear term, and then this class of models can be shown to converge to KPZ.

On Wednesday, Tom Alberts gave everyone an hour long overview of how the weakly asymmetric scalings of polymers and simple exclusion work, and described the localization of the polymer paths.

Another group took up the question of path localization in the random polymer and weakly asymmetric random polymers. After an afternoon of work, there were several successful outcomes and a clear picture emerged of exactly how localized the polymer is. Based on the conjecture that the spatial scale of fluctuations of KPZ type models is on the order of $t^{2/3}$ and is given by the Airy process, the group was able to use the new results about that process being local Brownian together with the behaviour of Brownian motion around a local maximum, to show that the continuum random polymer is localized in a tube of width order one around a special favourite path.

On Friday, Alison Etheridge gave a very interesting talk in which she described a number of biological models which appear to have KPZ scalings. The basic idea is that spatial population structure is dominated by large extinction, recolonization events. The species boundaries appear to grow as KPZ. These models are similar to stochastic reaction-diffusion equations in two dimensions, as well as simplified models like the Eden and Richardson models. These models do not have simply defined height functions and there was some discussion about how one would phrase the universality result, or if one could have weakly asymmetric limits in this context.