

# THE KADISON-SINGER PROBLEM

organized by

Pete Casazza, Richard Kadison, and David Larson

Workshop Summary

## *The “AIM” of the Kadison-Singer Conference*

The Kadison-Singer Problem (KS) grew out of mathematical Physics in the 1950's. It was formally stated by Kadison and Singer in 1955. By 1965 all ideas for the problem had been exhausted and the problem went dormant until 1980 when J. Anderson brought the subject back to life by introducing the idea of “paving” and showed that - what is now known as “The Paving Conjecture” - is equivalent to KS. This significant result generated a lot of excitement and research in the area. Bourgain and Tzafriri wrote several papers on paving. Berman, Halpern, Kaftal and Weiss wrote a series of papers on paving Laurant Operators and Akemann and Anderson wrote a Memoir of the AMS on this. But by 1991 the paving idea had run out and the subject went dormant again. Also, since Bourgain had worked very hard on the problem without success, people realized how really difficult the problem was and perhaps were a little discouraged.

In 2006, Casazza and Tremain - to the surprise of the scientific community - showed that KS was not just a specialized problem in one area of mathematics. But rather, KS is equivalent to fundamental unsolved problems in a dozen different areas of research in pure mathematics, applied mathematics and engineering including Banach space theory, Hilbert Space Theory, Frame Theory, Harmonic Analysis, Sampling Theory, Signal Processing, Sparse Representation Theory and more. Now, people from many different areas of research who knew little or nothing about one another have discovered that they have all been working on the same research problem for years. The idea of the AIM conference on the Kadison-Singer Problem was to bring the experts together from the many diverse research areas involved in KS to begin the process of cross fertilization of ideas. The goal was to bring everyone up to speed on the full scope of the problem and its many connections to areas of research different from their own.

## *The Program*

We had six one hour talks scheduled formally: 1. Casazza spoke on one of the most promising directions - paving projections with  $1/2$  on the diagonal; 2. Paulsen spoke on several new ideas from the  $C^*$ -algebra direction; 3. Akemann spoke on some of his old conjectures and some new ones related to them and what he knew about all this. 4. Nikolskii gave a wonderful talk showing how this problem relates to reproducing kernel Hilbert spaces and a long list of open problems there. This was all new to the audience since it went far beyond the Casazza/Tremain papers. 5. Weiss discussed a sophisticated computer program for paving small operators. This turned out to be of such broad interest we had him give a second talk on it. 6. Kadison discussed how this problem grew out of mathematical Physics

and quantum mechanics. He also showed in detail why he and Singer were convinced that the problem would have a negative answer.

### *The Research Groups*

At the workshop, there were 6 distinct research groups working on various aspects of KS. One was in  $C^*$ -Algebras run by Akemann and Paulsen. The idea here was that after Anderson introduced the paving conjecture, work on the  $C^*$ -algebra forms of KS came to a standstill. They felt that it was a mistake to abandon the original approaches to the problem and presented them in their group as well as formulating new versions of KS from this direction. There was a group run by Casazza and Larson concerning paving projections. Casazza gave a talk on work in progress with Edidin, Kalra and Paulsen, showing that KS is equivalent to paving projections with  $1/2$ 's on the diagonal. It was also shown that this class is not two pavaible. This group worked on producing some weaker forms of KS as well as examining how we might be able to produce a class of projections which would yield a counter-example to KS. Another research group was run by Johnson on the Bourgain-Tzafriri Conjecture (which is now equivalent to KS). This group showed that the Feichtinger Conjecture (which now is also equivalent to KS) holds probabilistically. This group also looked at approaches to generalizing a construction of Bourgain/Tzafriri to try to solve KS. Another group worked on the Feichtinger Conjecture and generic frames. This group also looked into the connections to the Rado-Horn Theorem and to Transversals and Matroid Partitions due to Edmonds and Fulkerson. Weiss had done extensive work on developing computer programs which combined with "theory" could quickly check the paving constants for small matrices. This work generated quite a lot of interest and so he gave a second talk on this as well as running a group session on it. Nikolskii gave a powerful lecture on how KS (in particular the weak Feichtinger Conjecture) relates to important questions in Harmonic Analysis and in general Reproducing Kernel Hilbert Spaces. In particular, he related it to questions on Carleson Measures, BMO, Besov spaces, work of Debranges and much more. Since the main point was to get people to interact, we pushed people into moving between groups to learn all the other aspects of KS. We have the groups give reports to the whole workshop on a regular basis. This worked quite well as we kept asking at the beginning of the sessions if everyone had seen all they needed to see from each of the groups and if not we quickly organized opportunities for them to get involved. Akemann, Weaver, Orr, Casazza, Paulsen, Kalra, and Weiss provided various private notes on KS for the participants.

*Collaborations* There were many collaborations started at the workshop. Larson and Han produced some results beginning a joint project. Johnson, Schlumprecht and a small group began a collaboration on the Bourgain-Tzafriri form of KS and produced some initial results on it. Larson, Casazza, Han and Kalra worked on producing projections with  $1/2$ 's on the diagonal. Donoho, Tanner and Casazza worked on engineering forms of KS. Wojdylo, Weiss and Schlumprecht worked on producing the "bad pavers" for the computer program. Paulsen and Akemann did joint work on  $C^*$ -algebra forms of KS and were joined by Tanbay, Packer and Li. Balan, Landau and Casazza started a joint project on the frame theory forms of KS. Jorgenson, Olafsson, Wojdylo, Kadison and Tremain attended all of the working groups to get their foot in the door. Nikolski and Wick worked on Harmonic Analysis and related. Kutyniok, Speegle, Casazza worked on the Rado-Horn Theorem and its relationship to KS. Krishtal worked with the projection group and the computer group. Weber is working with two groups who are doing frame theory approaches to KS. Casazza and Tremain convinced

Akemann, Paulsen, Nikolskii, Johnson and Weiss to collaborate on a large project (designed to be done by next May) to produce a (approximately 150 page manuscript) on KS which will have all the recent results as well as putting all the old results into the new framework of what we know today. This will include doing the Bourgain/Tzafriri construction right this time so it is usable in the area, connecting all the Laurant Operator stuff to the extensive list of new applications brought up by Nikolski, the Computer work, and bringing back to life all the old and forgotten approaches from  $C^*$ -algebras as well as a long list of new approaches arising out of the meeting.

### *Beyond the Conference*

An important part of the workshop was to not just get people working together, but to keep the subject alive even if things slow down after the momentum of the workshop. So the whole group spent almost two hours designing their web page which we expect to keep up to date until the problem is solved. Brett Wick volunteered to run the KS web page. The web page will start with a discussion by Jorgensen and Kadison concerning how this problem grew out of mathematical physics. Inside there will be four sections - with perhaps more added later. Each section will post papers (or connecting links to papers) on KS which relate to this form of KS. Each section will have an introduction to this form of KS and perhaps even a set of notes posted which will allow someone to enter this topic. 1. (Overseen by Nikolski and Wick) Harmonic Analysis and KS. 2. Paving and KS (Casazza and Weiss). 3. Frames and KS (Casazza, Larson, Speegle, Han). 4. State Extensions and KS (Akemann and Paulsen). Perhaps others will be added later - such as Banach space theory and KS run by Johnson and Schlumprecht. All papers will go to Casazza who will then forward them to the appropriate section. We will not "referee" these papers but just post them most of the time. The only "refereeing" to be done will be for "problems" submitted to be sure they are important enough that the answer will noticeably affect KS and be publishable. Otherwise, we may provide a link to the problems. We will have a statement on the page that problems on the web page have passed our test. There will be a mailing list for KS which anyone from the outside can join. There will also be an internal "blog site" open only to the people who attended the conference so they can talk to one another.

The bottom line here is that if this problem is solved in the foreseeable future, it will most likely be as a result of this meeting: the bonds formed during it, the new directions brought forth from the meeting, the enthusiasm generated for the problem and the web page designed to keep the problem alive this time. Perhaps this was best summarized by G. Weiss at the end of his talk: "Pete has opened the coffin. Now we have to decide what we are going to do about it."