Hypergeometric motives

organized by
Henri Cohen and Fernando Rodriguez Villegas

Workshop Summary

This meeting had a very specific goal: better understand the theory of hypergeometric motives (abbreviated HM) from many points of view, both theoretical, algorithmic, and computational.

HM were introduced some time ago by N. Katz, but the precise formulation, which is now accepted to be the correct one, was given by F. Rodriguez–Villegas, the main organiser of the meeting, after a previous meeting in Benasque (Spain) in 2009. These formulations and some computations, again both theoretical and practical, were explained in the first talks on Monday by FRV and H. Cohen, second organizer of the meeting.

One of the main interests of studying HM's is that they provide an infinite supply of *interesting and natural* L-functions of higher order, which are *easy to compute*, at least at the so-called *good primes*.

After the Monday talks and during the standard discussion needed to split into a number of small groups, it clearly appeared that there were the following avenues of research (given condensed, after the first days of work):

(1) Find recipes and algorithms to compute the Euler factors at the *bad but tame primes*, and the tame part of the conductor.
(2) Perform "reverse engineering" on HM's, in order to find some model of the algebraic variety corresponding to them.
(3) Produce large *databases* of HM's, so as to be able to formulate conjectures by studying them, and also for inclusion into the LMFDB database.
(4) On a purely computational side, study the algorithms for computing the Euler factors at the good primes and the inverse Mellin transforms of the gamma factors which occur.
(5) Use purely *analytic techniques* (such as approximate functional equations or similar) to find by *linear algebra* or successive approximations the unknown bad Euler factors and conductor.
(6) Use purely *algebraic* techniques such as *twisting* by Dirichlet characters to find the bad Euler factors.
(7) Try to recognize known *Siegel modular forms* whose L-function is the same as that of HM's.

The workshop enabled us to make considerable progress on some of these subjects, and as a whole, to better understand the issues involved. Since I (HC) had already worked on some of the implementation issues (Euler factors at good primes using the Gross–Koblitz formula and some inverse Mellin transforms), a slightly unexpected turn of events was that several of the small groups needed my program to be able to work efficiently. Since my

program was written for my own use and not documented, I urgently rewrote part of the program and made a small computer demonstration of its use.

On the different subjects above:

• Tame primes and tame part of the conductor: two groups worked on this. One more theoretically oriented was headed by D. Roberts, and the second dealing with implementation issues was composed of M. Vlasenko, B. Allombert, HC, and partly with F. Stromberg and G. Tornaria, and both groups were helped by FRV. We now have a working program which computes the tame part of the conductor, and the degree and factorizations of the Euler factors at the tame primes. It is expected that further work with D. Roberts will in fact give the Euler factors precisely, but the work already done narrows down considerably the naive search which was performed before for the bad factors.

• Databases: we (HC et al) were hoping to have large databases available by the end of the workshop, but the necessity of constant human intervention at this stage of the project prevented this, although it is a reasonable goal for the near future. We do have small databases computed by HC, G. Tornaria, and B. Allombert.

• Reverse engineering. This group was headed by F. Beukers, and its object of study was the close link between reverse engineering and A-hypergeometric functions. In a preprint which can be considered as the group report, Beukers has explained very clearly how the equations of the varieties can be obtained in a completelely systematic way, although his method does not always lead to the simplest equations. Further work of a combinatorial nature is needed on this.

• Algorithms for the good primes: before the workshop, HC showed that of the existing algorithms, the use of the p-adic Gross–Koblitz formula, especially when used modulo p^2 when possible, was by far the most efficient method to computed the good Euler factors.

• Algorithms for inverse Mellin transforms: although there do exist remarkable algorithms and implementations by T. Dokshitser, it seems that the simple use of shifting to $-\infty$ the line of complex integration is sufficient for actual computation of our L-functions.

• Algebraic techniques and twisting: M. Watkins and G. Tornaria worked on this, and to a lesser extent the group of D. Roberts. Experimental and theoretical evidence points to the existence for each motive of a Hecke Grössencharakter which gives the Euler factors at the tame primes. This is especially important when the tame prime becomes good, since "guessing" the correct Euler factor in that case is painful. Watkins made considerable progress on this, but much work remains to be done to determine precisely the explicit form of the Hecke character.

• Analytic techniques: group headed by D. Farmer, with S. Koutsiolikas and P. Molin. It was hoped initially that the methods used by Farmer et al. in other contexts would enable to compute the conductor and Euler factors at the bad primes, both tame and wild. These methods only led to finding lower bounds on the conductor, and were a little disappointing. However, towards the end of the workshop, a rather simple-minded idea proposed by K. Khury-Makhdisi, D. Zagier and others has seemed to give spectacular results which, together with new algorithms for the tame primes, could well entirely solve the problem of automating the search for the bad Euler factors. Since this was achieved 1 hour before the end of the workshop, it is too soon to determine its applicability, but the existing experiments look very promising.

Conclusion and followup: as a whole, I would say that the combination of people having quite different backgrounds and emphasis led to considerable progress on the subject matter of the workshop. As a concrete followup, we are seriously considering organizing a small workgroup of 4 or 5 people in places to be determined, meeting every 6 months to a year, to finish the work on tame primes, the present group being HC, FRV, D. Roberts, M. Vlasenko, and perhaps A. Mellit (who was not present at the workshop).