Free Analysis

organized by
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Workshop Summary

The idea behind the name of the workshop — free analysis — came from the realization that a number of common analytical problems about the L^2 theory of free difference quotient derivations are impeding progress in several areas, related to free probability theory. These areas include: free entropy (the analog of entropy in free probability); theory of L^2 Betti numbers for von Neumann algebras; stochastic calculus underlying random multi-matrix models; and large deviation bounds for random multi-matrix models; applications of free probability to von Neumann algebras of free groups.

Thus the main concentration of the workshop was on formulation, analysis and discussion of free analysis problems. The aim was an exploration involving: identification of the general problems; specific necessary conditions; possible sources of counterexamples; relating the different approaches being tried in the various contexts; and analysis of consequences the positive solutions would entail.

The workshop brought together a very diverse group of almost 30 participants with backgrounds as varied as classical probability theory, operator algebras, single operator theory and combinatorics. Therefore, another aim of the workshop was to encourage cross-fertilization of the various areas, sharing and explaining the specific notions and concepts as they arise in the various areas.

Free analysis problems are hard, but their occurrence in so many fields makes them unavoidable. We could not expect that these problems would be solved over the course of five days. What we hoped, however, and which, in our opinion, has happened, was to make sure that each participant leaves with a clear picture of how her or his work and the specific analytical problems relate and connect to those faced by others.

The format of the workshop called for starting the day with motivating talks (with generous time for questions), followed by discussion and problem sessions in the afternoon (these occasionally ran in parallel). The aim of the discussion sessions was to give ample opportunities for questions and discussion of possible conjectures and methods. One of the fundamental difficulties surrounding free analysis questions is the novelty of their highly non-commutative setting. We do not have much intuition to go on, other than the general pessimistic experience that "anything is possible" when one deals with non-commuting operators further compounded by taking operator limits. Thus it is extremely important to understand the correct conditions that are necessary for positive answers and results. One example of this, which was discussed a lot during the conference, relates to the question of discreteness and finite multiplicity of the spectrum of quantum Dirichlet forms. It was pointed out that such assumptions entail restrictions on the ambient von Neumann algebra generated by the non-commuting variables (the so-called Haagerup property). There was a lot of discussion of baisc inequalities (such as free Poincare inequality, free log-Sobolev

inequality, free hypercontractivity inequalities, etc.) and their consequences. For instance, it was possible to prove discreteness of spectrum of a quantum Dirichlet form arising from a matrix algebra perturbed by a semicircular family.

As mentioned above, it is important to collect situations and methods that work in hopes of being able to extend these. There are several important situations where questions similar to ones occurring in free analysis have been resolved. The first such situation involves the recent emergence in the work in control theory of Helton, Putinar and others of free semigroups and non-commutative polynomial calculus. Their results were new to most of the workshop participants and were received with great interest. Another situation involves the breakthrough work of U. Haagerup on smoothing techniques in free analysis. Using a kind of convolution with a free Cauchy law, he and his coworkers were able to obtain results on invariant subspaces of operators in finite von Neumann algebras. A number of questions were formulated based on his work; there is hope that some smoothing techniques of this kind may resolve deep problems surrounding the theory of free entropy. The third instance involves a recent discovery of N. Ozawa of applications of group boundaries to the study of structure of von Neumann algebras, including free group factors.

The workshop has succeeded in formulating a lengthy list of problems. Some of them appear to be very hard, but the motivation for asking these questions explains their value. It is clear that the workshop has opened the way for new results at the interface of the various directions connected to free analysis. In a way, the workshop has put the spotlight on free analysis problems and made everyone aware of their importance — not just in their subject, but in others' as well.