Open Problems in Frame Theory/Phase Retrieval
Prepared by John Haas

Introduction

The following compilation of open problems was collected during the "Frame Theory Intersects Geometry" conference, hosted by AIM, in Palo Alto, CA, in August 2013.

1 Notation

Given a frame $F = \{f_i\}$, we declare that $M :=$"the dimension of the ambient Euclidean space" and $N :=$ "the number of vectors in the frame." Such a frame will sometimes be referred to as an (N, M)-frame.

For problems 4 and 5, the function $A : \mathbb{C}^M/\mathbb{T} \to \mathbb{R}^N$ is defined by $A([x]) = (|\langle x, f_k\rangle|^2)_{k=1}^N$ for any choice $x \in [x]$. This function is well-defined, since $(|\langle \omega x, f_k\rangle|^2)_{k=1}^N = (|\langle x, f_k\rangle|^2)_{k=1}^N$ for any $\omega \in \mathbb{T}$.

We clarify the following acronyms:
ETF - equiangular tight frame
UNTF - unit norm tight frame

2 Problems

1. Zauner's Conjecture - Complex ETF's consisting of $N = M^2$ vectors exist for all $M \in \mathbb{N}$.
2. The Hadamard Conjecture - A Hadamard matrix of order $4k$ exists for every $k \in \mathbb{N}$.
3. The Paulsen Problem - If a frame is ϵ-close to being tight and ϵ-close to being equal-norm, then how far is it from being both equal-norm and tight?
4. Dustin Mixon: Does $N < 4M - 4$ imply that $A : \mathbb{C}^M/\mathbb{T} \to \mathbb{R}^N$ is not injective?
5. Dan Edidin: At the transition around $N = 4M - 4$, are there examples where the set of frames satisfying injectivity for $A : \mathbb{C}^M/\mathbb{T} \to \mathbb{R}^N$ is open, but such that the complement has positive measure?

6. Dan Edidin: If $B : \mathbb{P}^{M-1}/\mathbb{T} \rightarrow, [x] \mapsto (|\langle x, f_k\rangle|^2)_{k=1}^N$ is injective, what can can be concluded, if anything, about phase retrieval?

7. Bernhard Bodmann: Find N so that we have a quantitative measure for stability.

8. Dustin Mixon: Find N so that we have a quantitative measure for computational efficiency.

9. Ferenc Szollosi: Determine the maximum number of equiangular lines possible in $\mathbb{R}^{14}$. It is known that this number is 28, 29, or 30.

10. Ferenc Szollosi: Give a full algebraic classification of all complex ETF's of small parameters M and N.

11. Matt Fickus: Do real equi-modular ETF's exist (ie "± 1 matrices") whose redundancy is not approximately 2?

12. Ferenc Szollosi: Can we construct ETF's from nonabelian difference sets?

13. Matt Fickus: Are there integrality conditions on the dimensions for the existence of complex equiangular frames?

14. Matt Fickus: Can we find an elementary proof showing that, for any $2(M-2)$ 2-D planes in $\mathbb{R}^M$, there exists another 2-D plane that has principle angle of $\pi/2$ with each of the original planes?

15. Dustin Mixon: If we fix N and increase M, does the worst case coherence strictly decrease?

16. Matt Fickus: What is the threshold to know we are in the well of a global minimizer when we optimize over the UNTF's with cost function $U(F) = \max_{m \neq n} |\langle \varphi_n, \varphi_m\rangle|^2$?

17. Zhiqiang Xu: Suppose $x_0 \in \mathbb{C}^M$ and that $F = \{f_j\}_{j=1}^N$ is a frame in $\mathbb{C}^M$. Furthermore, suppose that X_0 is k-sparse (ie $\|x_0\|_0 \leq k$). Now set $b_j := |\langle f_j, x_0\rangle|, j = 1, 2, ..., N$. Then consider the following recovery problems (up to global phase factor):

    (a) What is the minimum N for which one can recover x_0 uniquely for b_j, $j = 1, 2, ..., N$.

    (b) What is the minimum N such that one can recover x_0 by solving the l_1 minimization:

    $$\min \|x\|_1, \ \ s.t. \ \ |\langle f_j, x\rangle| = b_j, \ j = 1, 2, ...N.$$

18. Boumediene Et-Taoui: Let $v(n, r, \mathbb{K})$ denote the maximum possible number of equi-isoclinic n-planes that can be imbedded in $\mathbb{K}^r$, where $\mathbb{K} = \mathbb{R}, \mathbb{C}$, or $\mathbb{H}$.

    (a) For any $r \geq 4$, find $v(2, 2r, \mathbb{R})$.

    (b) Find $v(2, 7, \mathbb{R})$.

    (c) For any $r \geq 3$, find $v(3, 3r, \mathbb{R})$.

    (d) Find the list of all regular v-tuples in $G(4, 4v)$ whose symmetry groups are isomorphic to the symmetry group S_v.

    (e) Find $v(3, 3r, \mathbb{K})$, where $\mathbb{K} = \mathbb{C}$ or $\mathbb{H}$.

(f) Find all $(2M, M)$ complex equiangular tight frames.

19. Emily King: Let U denote the space of UNTF's with dimensions (N, M). If it is known that complex ETF's do not exist in (N, M), then what is the best one can do? In particular, optimize:

$$\min_{F \in U} \max_{i \neq j} |\langle f_i, f_j \rangle|.$$

20. Do complex ETF's exist for the parameters $M = 3$, $N = 8$?

21. Construct a table/catalogue which has entries corresponding to the different pairs (N, M) for small N and M, indicating whether complex equiangular frames are known to exist, not to exist, or if the problem remains open. If the problem is settled, then the table should also describe by what methods the information is ascertained. For example, the entry corresponding to $(7, 3)$ would indicate that they exist due to construction via difference sets. On the other hand, the entry for $(8, 3)$ would indicate the question remains open.