Cohomology and representation theory for finite Lie groups

organized by
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Workshop Summary

1 Introduction

As stated in the original announcement, a major workshop aim was to bring together two groups: (1) those with research interests related to the modular representation theory and cohomology of finite groups of Lie type; and (2) experts in computation and applications whose experience may or may not lie directly in the field, but whose knowledge could add depth to the pool of ideas for the first group. The AIM workshop ran from June 25–29 2007 with 19 faculty and 5 graduate students in attendance who well represented both groups (1) and (2).

Some principal targets for these groups at the workshop were to assess the current state of past, present, and potential uses for computation in the field, and to identify, assemble, and discuss a list of problems for which progress might significantly hinge on intensified computation. Many theoretical practitioners (and potential researchers) lack an awareness of computational methods as a tool for theoretical advances, so taking steps to address various obstacles to employing computational methods constituted another important goal.

2 Progress Toward Workshop Goals

The workshop was organized, at least initially, around several expository talks (in chronological order):

• Bhama Srinivasan, "ℓ-modular representations of finite reductive groups."

• Zongzhu Lin, "Introduction to Kazhdan-Lusztig polynomials."

• Jon Carlson, "Calculating cohomology."

• Frank Lübeck, "Computing (with) characters and representations."

• Alfred Noël, "The Atlas of Lie groups and representations: Character Table."

• Robert Guralnick, "Cohomology of finite groups."

• Leonard Scott, "Reduced standard modules and cohomology."

• Bhama Srinivasan and Peter Webb, "Categorification."

• David Hemmer, "An introduction to the cohomology and modular representation theory of the symmetric group."

Starting on day one, with these talks as backdrops and motivation, participants gathered, in groups both small and large, to assemble and investigate a list of "problems" for consideration during (and after the workshop). These problems represented both information sought by group members for their own understanding, and specific research problems thought to be of importance for the field at large. In preparing and reviewing draft lists, participants furthermore considered (in varying degrees) factors such as prior and potential uses of computational methods on a given topic, as well as the degree of difficulty of the problems, and/or their adaptability to projects for students or groups of individuals at differing institutions.

The talks by J. Carlson, F. Lübeck, and A. Noël directly addressed computational methods as applied to some key theoretical issues. The methods utilized therein involved a variety of software packages and programs with high levels of potential use for problems in modular representation theory (and mathematics at large). For example, J. Carlson described his work using the software package Magma, F. Lübeck concentrated on applications of GAP, and A. Noël discussed progress by the Atlas of Lie Groups project on the computation of Kazhdan-Lusztig polynomials. Subsequently, the workshop broke into small groups involved in various "demos" (by Lübeck, D. Roozemond, and C. Bendel) exploring capabilities of specific packages in more detail, including their applicability to specific problems of interest to modular representation theorists at the workshop.

3 Theoretical Horizons Driven by Workshop

In this section, we concentrate on some promising avenues opened (or widened) by the workshop activities. As apparent from the length of the current draft of the workshop Problem and Discussion List, there were many more topics broached than the four we raise below, so we expect there to be more outcomes in the future.

3.1 Support varieties

Over the past 25 years, the study of cohomological support varieties has contributed greatly to advances in the modular cohomology and representation theory of finite groups and related objects. In the discussion of open problems, participants reaffirmed that a key challenge continues to be explicit computations of support varieties. For a reductive group G defined over an algebraically closed field $\overline{\mathbb{F}}_p$ of characteristic $p > 0$, let $F : G \to G$ be the standard Frobenius morphism. A close relationship exists between the representation theory of the finite subgroups G^{F^r} of F^r-fixed points and the representation theory of the infinitesimal subgroups (Frobenius kernels) $G_r := \ker(F^r)$. Thus, seeking to develop more direct connections between the support varieties of the finite groups of Lie type G^{F^r} and the Frobenius kernels G_r is also a major goal.

For finite groups of Lie type (i.e., the G^{F^r} above), making direct computations of their support varieties involves, as a first step, understanding the maximal elementary abelian subgroups of such groups. During the workshop, Á. Seress wrote a program using GAP to begin exploring this question for general linear groups. Optimistically, further computations will suggest general patterns that can be proved theoretically.

For Frobenius kernels (i.e., the G_r above), at a fundamental level, computing support varieties requires an understanding of pairwise commuting tuples (pairs, triples, ...) of nilpotent matrices. While the variety of nilpotent matrices is well understood, this is not the case for commuting n-tuples (especially as the number n increases). R. Guralnick updated a large group of interested participants on the current knowledge about these varieties;

computing them seems to be a very hard problem, hence even explicit computations for small rank cases would be useful.

A related computational issue (in both contexts above) is the ability to work with certain fundamental modules in a computer algebra system. For example, an important family of modules to consider is formed by the Weyl modules for the underlying algebraic group G. With the help of F. Lübeck, participants were able to learn about certain programs within GAP that can be used to describe the structure of Weyl modules in positive characteristic. While this functionality exists (in the form of F. Lübeck's programs), it is not conveniently accessible to the general public[1]. This is just one example of how bringing theoreticians and computationalists together created the opportunity for computationalists to learn what tools the theoreticians need. It also points out a potential growing need: how researchers and the field at large can provide effective and substantial assistance for software developers to not only create new programs to meet the field's needs, but to implement (or even just post) programs and results they already have, in a form accessible for a less experienced public.

A group of participants began investigations into computing support varieties for Weyl modules for higher Frobenius kernels $G_r, r > 1$. The answer is known for first Frobenius kernels G_1. For higher Frobenius kernels, essentially the only known results are for $G = SL_2$ in joint work of A. Suslin, E. Friedlander, and C. Bendel from the mid-1990s. Participants reviewed this work and were optimistic that some progress could be made for larger rank groups (e.g. SL_3) potentially with the aid of computational technology. More generally, the participants made a conjecture on the support varieties of certain Weyl modules[2].

3.2 Guralnick's Conjecture

During the workshop, R. Guralnick discussed several topics, including his 1984 conjecture that there exists a constant C with the following property: if G is any finite group and V is any faithful irreducible representation of G (over some field), then dim $H^1(G, V) \leq C$. Computations of cohomology groups, such as $H^1(G, V)$, have wide-ranging applications for the structure and representation theory of finite groups, and hence for other applications dependent upon this theory, such as the theory of finite automata[3]. Recently, Guralnick's conjecture has been studied by E. Cline, B. Parshall and L. Scott (CPS) who have proved that given a root system Φ, there is a constant $C(\Phi)$ bounding the generic cohomology dim $H^1_{\text{gen}}(G, V)$ whenever G is a finite group of Lie type arising from a semisimple algebraic group with root system Φ. Based on questions/comments by C. Bendel, R. Guralnick, and C. Pillen during the workshop on a presentation of this work by L. Scott, CPS have recently been able to extend their bound to include the actual cohomology $H^1(G, V)$ (not just the generic cohomology). A closely related topic involves the behavior of the leading coefficient $\mu(x, y)$ of Kazhdan-Lusztig polynomials, a very interesting computational problem. Of course, the specific determination of Kazhdan-Lusztig polynomials was a major topic of the workshop,

[1]That is, no online macro for making the relevant calculations exists for public use; access to these programs would be only through F. Lübeck's generous offer to run them himself, for any workshop participant who needs these calculations and sends him their starting data.

[2]See the "AIM Conjecture" in the section "Support Varieties" of the current Problem and Discussion List for a statement and more details.

[3]For more details, please see §3 of of the introductory notes "Irreducible modular representations of finite and algebraic groups" prepared for the workshop by C. Drupieski and T. Hodge, based on lectures by L. Scott.

with discussions and contributions by many participants, particularly Z. Lin and L. Scott. There seems to be some optimism that, taken all together, a final answer (or counterexample) to Guralnick's conjecture may be within sight.

3.3 Koszul Structures

A Koszul structure on a positively graded algebra is a strong condition which (remarkably) occurs quite often in Lie theory. For example, the principal block of the category $\mathcal{O}$ associated to a complex semisimple Lie algebra is Koszul. Unfortunately, most of these positivity results work only in characteristic zero, since they require deep geometric results on perverse sheaves. In prime characteristic, it has long been speculated that Schur algebras might have a Koszul structure. Such a fact would have very strong consequences in modular representation theory. During the workshop, a start was made on this problem by J. Carlson, D. Hemmer, B. Parshall, L. Scott, and L. Townsley. In this work, Carlson was able to use his computational expertise to explicitly describe the structure of the PIMS for the basic algebra associated to Schur algebras in small characteristic[4]. When the conditions of the James conjecture are met, it seems that the Schur algebras considered are Koszul. Otherwise, the algebras are not Koszul. These results need to be further checked, but the exciting fact is that they give the first evidence that examples can actually be computed.

3.2 Symmetric Groups

Going back to Cayley's theorem, the symmetric groups Σ_d, $d \in \mathbb{N}$, constitute essential examples of groups for which computing their representation theory is of great significance in a multitude of applications. In this light, the Specht modules S^λ (one for each partition λ of d) are of particular importance; for example, for the representation theory of a symmetric group over the complex numbers, all irreducible modules arise as Specht modules, and their dimensions are known via nice combinatorial descriptions involving standard Young tableaux (or the celebrated hook length formula). Over a field of positive characteristic p, the Specht modules need no longer be irreducible. However, the irreducibles arise as heads (or socles) of appropriate Specht modules (the S^λ for which λ is 'p-restricted'); in general, however, determining the dimension of the irreducibles or the socle of S^λ for λ arbitrary are open questions. Determining various relationships between irreducibles and Specht modules (e.g., decomposition numbers) and computing cohomology (Ext) groups for Specht and irreducible modules are other essential problems, discussed during the workshop, for which concrete computational approaches using modern software may bring some insight.

Under appropriate hypotheses on $d, n \in \mathbb{N}$, the representation theory of Σ_d is intimately linked with the representation theory of the general linear group GL_n (as an algebraic group) by the Schur-Weyl theory. This connection leads to many open questions associated to comparisons of the symmetric group and general linear group theories, such as questions of comparing their cohomology. Moreover, the representation theories of Schur algebras and q-Schur algebras, which give information about representations of GL_n[5], also come into play. Hence, the discussion of computations for the Schur algebra in § has implications for the representation theory of Σ_d as well. In formulating these and other questions, D. Hemmer's

[4]Computations appear at http://www.math.uga.edu/∼jfc/schur.html, with an acknowledgement to AIM.

[5]As detailed in §4 of the introductory notes "Irreducible modular representations of finite and algebraic groups".

leadership and background talk listing a host of additional open problems were particularly instrumental. Through talks by B. Srinivasan and P. Webb, participants also began to learn of 'categorification', a key idea in the very recent proof by Chuang-Rouquier of the Broué conjecture for symmetric groups which may have important implications for the describing characteristic representation theory of finite groups of Lie type.

4 Conclusion

The workshop addressed the original goals of educating theoreticians on computational methods, bringing together researchers from differing subfields to share their expertise, and creating a list of problems of import which may be attacked computationally. To the organizers, an additional, but very significant, outcome of the workshop was the promotion within the field of new, successful modes for teaching, learning, and researching, arising from the implementation of the AIM workshop format itself and from lively discussions surrounding the engaging presentations by B. Boe, B. Cooper, and C. Wright about the VIGRE program at the University of Georgia (UGA). Particularly noteworthy was the potential of the formats embodied by these two structures to meaningfully engage students, as well as researchers with varying levels of experience in the field, in high-quality discussions and collaborative research (such as transpired during our own workshop experience).

The organizers hope that the sense of camaraderie developed between the workshop attendees and an AIM-style ripple effect on educational and scholarly efforts in the field will continue in the future. Specifically, we hope for the AIM website started for this workshop to evolve into an important resource for mathematicians interested in computational methods in modular representation theory. In conjunction with other participants, we are reviewing options for pursuing possible follow-up grant opportunities, as in discussions initiated during the workshop. Individual participants have spoken to us with excitement about their AIM experiences and of their intentions to try out some AIM and UGA VIGRE-motivated approaches at their own institutions. In a related direction, some of us plan to organize a Special Session of the AMS in the October 2008 meeting in Kalamazoo along the lines of the AIM workshop (but on a smaller scale). Through these activities and the AIM workshop itself, the a priori goal of fostering collaborative relationships between researchers and potential researchers at a variety of institutions and with varying levels of expertise and career development has been/will have been met and expanded.

Of the original list of workshop goals, two were not substantively addressed during the workshop: hurdling many additional barriers to employing computational methods (other than a lack of expertise and information[6]), and generating more awareness of applied issues. Due to the constitution of the final group of participants, the latter was an issue the organizers set aside. For the former, this was indeed a topic of discussion during the workshop, and is something the organizers hope to address in future activities and through the website mentioned above.

[6] As previously noted in this report, these two principal obstructions were significantly addressed, through demos, focused discussions, and other contacts between theoreticians and computationalists.