The Cuntz semigroup

organized by
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Workshop Summary

The object of this workshop – the Cuntz semigroup – was defined over thirty years ago, but was recently proved to be isomorphic to another semigroup, one defined completely differently. As such, the first day was devoted to bringing all participants up to speed on the two pictures of the Cuntz semigroup, as well as basic techniques and results needed to work with these objects. In the morning there were two introductory lectures, one exploring the apparently different definitions and the connections between them, the other lecture outlining some important applications of Cuntz's semigroup. In the afternoon we organized three "ask an expert" groups and one "technical problems" group. In the former, an expert would field questions from participants less familiar with the group's topic, while the latter was reserved for the (relatively few) participants already on top of both definitions and all the technical results one needs to know in order to work in this area.

For the remainder of the week, afternoons were spent in working groups. Here's a summary of the main themes of discussion, as well as some of the progress made on particular problems.

(1) Answering a question of Cuntz, it was shown during the workshop that the Cuntz semigroup of a C*-algebra A is isomorphic to the Cuntz semigroup of an abelian C*-algebra if and only if A is Morita equivalent to an abelian algebra. In addition to being of independent interest, this nice result shows that some important C*-structure can be deduced from the Cuntz semigroup. Other results of this type had been known (e.g., recovering an algebra's ideal lattice from its Cuntz semigroup), but this was an unexpected addition to the known results.

(2) Partially answering another question of Cuntz, it was shown that the "open unit ball" of the Cuntz semigroup has a unique maximal element, in the simple case. Those working on this problem expressed confidence that, in general, the left kernel of a pure state defines a maximal element, however questions such as when this element is unique, remain open.

(3) Motivated by analysis of the Cuntz semigroup for so-called AH algebras, two groups made progress on the classification of these algebras in the simple real rank zero case. The first result is that unital simple AH algebras with unique tracial state, stable rank one, and infinitesimal-free K_0-group are classified up to isomorphism by their graded ordered K-theory, leading in particular to a complete description of their Cuntz semigroups. The second group replaced the tracial and K_0-group conditions with a restriction on bonding maps, and while they did not get classification, they were able to develop a previously untested line of attack—work is expected to continue beyond the conference.

(4) One fundamental question encountered when trying to use the Cuntz semigroup for classification is how to incorporate K_1-type data. (For example, there exist $A\mathbb{T}$ algebras with isomorphic Cuntz semigroups, but non-isomorphic K_1 groups.) Hence, as a first step toward a uniqueness theorem, it is natural to ask whether the Cuntz semigroup can be used to classify approximate unitary equivalence classes of unitaries. (The Cuntz semigroup is known to classify approximate unitary equivalence classes of self-adjoints, in many cases.) Though participants did not solve this problem in general, important progress was made in some nontrivial cases. For example, when the unitaries are assumed to be exponentials, or if the ambient algebra is the Jiang-Su algebra. It is hoped that this progress will lay the foundation for future collaboration, and we also note that the group working on this included both senior researchers and graduate students.

(5) One group of participants spent three days seeking the right definition of a bivariant version of Cuntz's semigroup. Though this was unsuccessful, several possibilities were definitely eliminated and it seems very likely that this work laid the groundwork for future collaboration. The group included several graduate students and it is hoped that future work will include them.

(6) A group which met three times during the week considered the question of whether elements of the Cuntz semigroup of a unital simple infinite-dimensional C*-algebra can be approximately divided by an arbitrary natural number. (A positive answer is known to have strong consequences for the classification of C*-algebras.) This is a very technically demanding and difficult question, and no major breakthroughs were achieved. There was, however, a new approach to the question, which will be tested in the coming months. It is based on the fact that the desired divisibility result holds in the real rank zero case, and the course of the proof can at least be attempted with projections replaced with positive elements.

(7) A final group met only on Friday, with an idea to prove that in fact the answer to the question in (6) is negative. They observed that as with vector bundles, there may be homotopy obstructions to extending positive operators over a closed subset of a metric space to the whole metric space unless one allows some variation in rank. It is hoped that an analysis of the homotopy type of the space of nonnegative definite matrices with rank between k and l will shed light on the magnitude of this rank variation. If it can be shown to be on the order of the dimension of the underlying space, then there is some hope that a counterexample to (6) can be found.

To facilitate future work on the Cuntz semigroup and improve the accessibility of the subject for graduate students and young researchers, an effort was made to collect the statements of a multitude of technical results that are scattered throughout the literature. Proofs, however, could not be collected on short notice, only references. One young participant proposed a solution: he offered to create a community edited webpage, in the style of Wikipedia, where basic facts related to the Cuntz semigroup can be added – with proofs. This is a great idea and the organizers will do their best to ensure that it comes to fruition.

In summary, the meeting was very successful. Many interesting problems were communicated to a wider audience, new problems were suggested, some problems were solved, and many others saw material progress.