Convex algebraic geometry, optimization and applications

organized by
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Workshop Summary

We start with a little bit of terminology. A linear pencil is a function $L : \mathbb{R}^g \to d \times d$ *symmetric matrices* of the form

$$L(x) := L_0 + L_1 x_1 + \cdots + L_g x_g$$

with the L_j being $d \times d$ symmetric matrices. A linear matrix inequality (LMI) is one of the form

$$L(x) \succeq 0$$

with $\succeq 0$ meaning $L(x) \succeq 0$ is positive semidefinite. LMIs have been one of the main successes and interests of optimization theory and numerics in the last 15 years. The study and algorithms for solution of LMIs is called semidefinite programming. A polyhedron is the set of solutions to ordinary linear inequalities and by analogy the set of solutions to an LMI is called a spectrahedron. That is, it is a set in $\mathbb{R}^g$ of the form $\mathcal{D}_L := \{x \in \mathbb{R}^g : \ L(x) \ is \ PSD \ \}$; obviously it is convex and semialgebraic. If a set $\mathcal{C}$ in $\mathbb{R}^g$ is a spectrahedron, it is said to have an LMI representation.

The following is an account of some of the group topic discussions which occurred each afternoon:

- **Dimensional differences of faces of nonnegative and SOS polynomials**
  The cone of nonnegative polynomials contains the cone of polynomials which are sums of squares of polynomials (SOS) properly. Their geometric structures are very different. It is known that there are significantly many more polynomials than sum of squares. This can be reflected by the dimensional differences between their faces. Their dimensions are different when there are nonnegative polynomials which are not SOS. These dimensional differences can bring valid inequalities to separate nonnegative polynomials from the SOS cone. The main research issue is how to get these separation inequalities efficiently.
- **SDP representation of convex sets and hulls**
  The main topic of this workshop is SDP representability issues. There are recent constructions by Lassere and Parrilo parameterizing convex semialgebraic sets as projections of linear matrix inequalities. Helton-Nie proved this happens when the boundaries are smooth and have positive curvatures. The main research issues now are how to handle the singularities and zero curvature. To resolve singularities, the standard techniques from algebraic geometry are potentially very useful, but the biggest trouble is how to preserve convexity. For the case of zero curvatures, the difficulty is how to represent nonnegative polynomials, which is one of the most difficult problems in polynomial optimization. A lot of trial approaches are discussed in the group discussion.

A related issue is how to represent the convex hull of varieties. When a variety is a curve parameterized rational polynomials, its convex hull has an easy SDP representation. However, for higher dimensional varieties, its lifted LMI is quite difficult to get. The main issue: there is no general SOS type representation for nonnegative polynomials.

- **Numerical methods in polynomial optimization**
The main discussion topic was how to solve large scale polynomial optimization. For SOS relaxations, if interior-point methods are used, only small problems could be solved. The reason is the resulting SDP from SOS relaxation is in the most difficult class of SDP problems, because the number of equality constraints is the square of matrix length in magnitude. For this type of SDP problem, the regularization type of methods like augmented lagrangian are very suitable. The memory requirement is very low. The main issues in this type methods are how to solve a subproblem efficiently. Since the performance of Conjugate Gradient (CG) iteration significantly effects the convergence of the whole algorithm, the preconditioning is important. The future research problems include: how to improve semismooth Newton method, find better preconditioners for CG iterations in solving Lasserre's relaxation, seek more clever line search techniques, find better ways to update parameters. Nie described his new conjugate gradient SOS solver and empirical finding that positive degree 6 polynomials in higher dimension are often not SOS.
- **Quadratic modules and tropical varieties**
The discussion group originated from issues raised by Salma Kulmann on quadratic modules when Bernd Sturmfels pointed out they were related to the rapidly moving are of tropical geometry. This group consisted of about a dozen of workshop participants, at least half of them were young researchers. We first recalled the definition of the locally convex topology on the polynomial ring, and reviewed Berg's theorem which establishes that the quadratic module of sums of squares (SOS for short) is closed. This theorem was generalized later by Kuhlmann-Marshall to a finitely generated quadratic module M that describes a cone with open interior, and let to the key notion of stable quadratic modules due to Powers-Scheiderer. A key lemma in the proof is the fact that if a polynomial $p(x)$ is nonnegative on a cone, then so is its leading term (homogeneous component of highest degree) $LT(p(x))$. Salma suggested that $LT(M) := \{LT(p(x)) : p \in M\}$ is an interesting object for investigation, possibly under other weighted degree or monomial orderings w. (This should also connect to the "preodering membership problem" studied by D. Augustin in her dissertation.) Graduate student Cynthia Vinzant has a first result in this direction. She showed that given an ideal I, if for some weighted degree w we have that $LT_w(I)$ is real radical, then SOS + I is stable. This is related to work of Tim Netzer (dissertation). B. Sturmfels explained in this context the notion of Groebner Fan of an ideal. This led to a lively discussion. The following problems were suggested for further study:
Problem 1: Study the Groebner Fan of a quadratic module (Kuhlmann).
Problem 2: Study the examples of stable preorderings of planar curves described by Scheiderer in light of Vinzant's result (Marshall).
Problem 3: Given the quadratic module generated by a single polynomial in 2 variables, decide whether it is stable or not (Marshall). Before parting, Bernd Sturmfels

gave us the definition of the tropical variety associated to an ideal, and provided some homework on this topic for the participants.

- **Degree Bounds for Positivstellensatz, boundary structure of Hyperbolic cones, derivative cones, extreme rays of hyperbolic cones**
  The discussion started with Markus Schweighofer describing the current state-of-the-art for the degree bounds for Positivstellensatz. Markus gave a sketch of some of the proofs and described the ingredients of the others. Then a discussion of the quality of these bounds and their limitations took place. A brainstorming activity on SDP relaxations and convex hulls lead the group into the subject of hyperbolic polynomials and their hyperbolicity cones (also called hyperbolic cones). Levent Tunçel first went over the structural results by Renegar involving the derivative cones and then discussed some newer, unpublished results expressing every hyperbolic cone precisely as the intersection of its derivative cones (where the intersection ranges over all interior points of the original hyperbolic cone). This result shows that the boundary structure of hyperbolic cones is completely determined by its derivative cones. The session ended with a discussion of hyperbolicity preserving operations on polynomials, strictly hyperbolic polynomials, some partial characterizations of the extreme rays of hyperbolic cones and finally, possible future research directions for improving our understanding of the boundary structure of hyperbolic cones.
- **Functional Analysis Issues**
  One session was attended by Salma Kuhlmann, Bill Helton, Victor Vinnikov, Mihai Putinar and graduate students Jeremy Greene, Martin Harrison, Chris Nelson, and Joules Nahas.
  We consider Salma's question of investigating the closure of the SOS polynomials in various locally convex topologies. These closures would then give a quantitative measure of how the topologies differ. In particular, the SOS polynomials are already closed in the finest locally convex topology; and, as it figures importantly in the work of Laserre, the closure in the L1 (on the coefficients) topology is the polynomials which are non-negative on the unit cube.
  The session began with Salma - with her background in model theory and semi-algebraic geometry - exposing the relevant objects and topologies to the functional analysis who made up the remainder of the group. We then generated several conjectures, including the guess that the closure of the SOS polynomials in g variables in the L^2 (on the coefficients) topology would yield exactly f in H^2 of the g-polydisc whose restriction to the cube of $[0, 1)^g$ are non-negative.
  We also heard reports from each of the graduate students at UCSB and UCSD and discussed possible future directions for their research. Martin Harrison and Joules Hahas developed a non-commutative version of Laserre relaxation; and Joules proved and used smoothness results for PDEs to established an improved version of Schmudgen's positivstellensatz for the Weyl algebra. Jeremy Greene has characterized non-commutative plurisubharmonic polynomials; Christopher Nelson has characterized all non-commutative harmonic polynomials.
  Another session looked at the basic question is what are the corners of a spectrahedron. That is, what is the smallest rank r_* of a matrix of the form $L(x)$ with $x \in \mathcal{D}_L := \{x : \ L(x) \succeq\}$? What ranks occur? These questions were featured in Bernd Sturmfel's talk which opened the conference. See slides of Sturmfels talk

posted on the AIM website for definitions pictures and context. Another issue is how might one possibly compute x_* minimizing rank $L(x)$. Rank minimization subject to convex constraints is an important practical but highly nonconvex problem. Maryam Fazel with Parrilo and Recht have a probabilistic analysis for minimizing *trace* $L(x)$ as a heuristic for finding an x_*, which is analyzed in a much more special case. This parallels the famous analysis by Candes and Tao. A session discussion speculated on a relaxation of her method (with Parrilo and Recht) dependent on a parameter ϵ, and started to see if one could prove probabilistic estimates of success by generalizing her methods. We looked at polytopes for starters.

In addition there was a session was on matrix completion problems arising from applications; notably one from statistics and one from engineering systems theory. These were suggested by Carolyn Uhler and Parikshit Shah graduate students in UCB Statistics Dept and MIT Engineering respectively. Any advance on the basic classical matrix completion problems would increase the range of results in these areas. Parikshit described a clever generalization via matrix completion of the crucial (to control) Finsler Lemma. Carolyn gave elegant results on covariance matrices. Also arising from the statistics application there is an interesting extra constraint one can add to the completion problem, thereby generating more open questions. The pure math question is:

We are given a matrix X of size $m \times n$ $(n < m)$ and an undirected graph G on m vertices. We denote by $A(G)$ a G-partial matrix, where the entry a_{ij} is defined if and only if $(i, j) \in E(G)$ or $i = j$. Under what conditions on the row-vectors of the matrix X (respectively on the angles between the row vectors of X) does the G-partial matrix $XX^T(G)$ have a PD completion?

- **Sum of Squares and Polynomial Convexity**

  This session focused on the interplay between the concept of sum of squares (SOS) decomposition and the question of deciding convexity of polynomials. As defined by Helton and Nie, a multivariate polynomial $f(x) = f(x_1, \ldots, x_n)$ is SOS-convex if its Hessian $\nabla^2 f(x)$ is an SOS-matrix; i.e., if $\nabla^2 f(x)$ can be factored as $\nabla^2 f(x) = M^T(x)M(x)$ with a possibly nonsquare polynomial matrix $M(x)$. This is a sufficient condition for convexity of polynomials that can be efficiently checked with semidefinite programming.

  Amir Ali Ahmadi an engineering graduate student at MIT, observed that there are other natural definitions that could be given for SOS-convexity. Namely, instead of working with the Hessian matrix, we can use the sum of squares relaxation on the inequality in the definition of convexity or in its first order characterization. He showed that all of these relaxations will be exactly equivalent. He then proceeded to prove that SOS-convexity is *not* a necessary condition for convexity of polynomials. He presented an explicit example of a trivariate homogeneous polynomial of degree eight that is convex but not SOS-convex.

  Finally, he presented some more recent results, where there is much room for further research and new contributions. We shall end this discussion by sharing with the reader two related open problems raised by Amir that might be of interest.

  After finding the first example of a polynomial that is convex but not SOS-convex, it is natural to ask in what degrees and dimensions there is a gap between convexity and SOS-convexity. Amirs recent results seem to suggest that there is a gap

between convexity and SOS-convexity *exactly* in situations where there is a gap between nonnegativity and SOS (as described in the so called "Hilbert table"). Is this a mere coincidence? Or is there a deeper connection between some algebraic/geometric aspects of convexity and positivity that we currently do not fully understand?
The second problem also has to do with the connection of convexity/SOS-convexity with nonnegativity/sum of squares. Is it true that every convex nonnegative polynomial is a sum of squares? Although one might see no reason for this to be true at first glance, Amir has sensible evidence to indicate that this could potentially be true. About 10 days after the AIM workshop Greg Blekherman, active in many of our AIM discussions, posted an asymptotic estimate implying the answer is no for high degree polynomials. http://front.math.ucdavis.edu/0910.0656.