Model Theory of Metric Structures

organized by
C. Ward Henson and Itaï Ben Yaacov

Workshop Summary

The first day

We began Monday morning with a lecture prepared by the organizers and delivered by Ben Yaacov, presenting the background on continuous logic and metric structures that would be needed during the week. This was designed to elicit lots of questions and discussion, and the end result seemed pretty successful. This event lasted most of the morning due to the many (very fruitful) interruptions, and its lively, interactive nature enhanced the cohesiveness of the group right from the start. A text version of the slides used by Ben Yaacov in this session will be made available at the workshop website. References for this lecture were the preprints [BBHU] and [BU], which were available to participants before the workshop; printed copies were made available by AIM staff Monday morning.

The afternoon ended up being entirely devoted to a full-group discussion of topics that we might concentrate on during the rest of the week, moderated by Anand Pillay. A separate document prepared by Alex Berenstein (our web liason) lists the many topics that were proposed – there are 29 items in the list! This is available at the workshop website.

Overnight the organizers distilled and shaped this list, reducing it to 12 items, each of which would be offered Tuesday morning as a possible topic on which a work group might concentrate.

Work groups

Concentration topics were selected Tuesday morning, in a general session that lasted about two hours. Ultimately each person was required to choose a single topic. Six work groups resulted from this process, and they began work immediately afterward. There was moderate change in the structure of these groups during the week. Apart from decisions by a handful of people to change groups, the main changes were these: two of the original work groups ("weak metric structures" and "probability structures") decided to merge, because several people wanted to attend both groups and because it was felt that the two topics had some overlap. Also, a new work group on "non-commutative probability spaces" was formed Wednesday morning, after the late arrival of Marius Junge, who was a key person for this topic.

Every day from Tuesday to Friday there were reports from each work group, of length appropriate to what they had been doing. Several times this led to good advice to a group from the general audience, as well as to an exchange of ideas between groups.

We will ensure that there is eventually a written report from each group. (All six reports are underway, with four of them being essentially completed.)

The work groups and their memberships were as follows:

- *Asymptotic cones*: Sylvia Carlisle and Françoise Point.
- *Stable groups in metric structures*: Paul Baginski, Alex Berenstein, Alice Medvedev and Frank Wagner.
- *Definable sets*: Uri Andrews, Clifton Ealy, C. Ward Henson, Åsa Hirvonen, Julien Melleray, Ludomir Newelski and Slawomir Solecki.
- *Weak metric structures and probability structures*: Itaï Ben Yaacov, H. Jerome Keisler, Alexei Kolesnikov, Maryanthe Malliaris, Anand Pillay, David Ross, Thomas Scanlon and Shichang Song.
- *Definable norms and stable Banach spaces*: José Iovino, William Johnson, Carlos Ortiz, Yves Raynaud, Christian Rosendal, Alex Usvyatsov and Monica VanDieren.
- *Non-commutative probability spaces*: Itaï Ben Yaacov, C. Ward Henson, Marius Junge and Yves Raynaud.

Impromptu events

With the exception of the Monday morning lecture, all presentations were decided on by popular demand during the workshop, arising from general discussion about which ones would be valuable.

Ben Yaacov gave a lecture on perturbations of metric structures, mainly based on his preprint [Ben]. This is a new concept that applies naturally within continuous logic (but is vacuous in ordinary model theory) and it seems to be of growing importance. A text version of the slides used during this lecture will be made available at the workshop website.

Also, we pressed several participants into service to present some important examples that kept coming up in discussions but were not familiar to everyone. We organized three "examples" sessions to accommodate these presentations. They were very spontaneous, given without much preparation, and provided information that was used in several of the working groups later in the week.

The examples presented were:

- Unit balls of Hilbert spaces with generic unitary (Alex Berenstein).
- $L_p(\mu)$ Banach lattices and Nakano spaces (Ward Henson).
- Asymptotic cones of finitely generated groups (Françoise Point and Sylvia Carlisle).
- Tsirelson's Banach space (Bill Johnson).
- Urysohn's metric space (Julien Melleray).

New collaborations and interactions

Non-commutative probability.
The group on non-commutative probability spaces (Ben Yaacov, Henson, Junge and Raynaud) will continue collaborating on the topic. There had already been some collaboration on other topics between Henson and Ben Yaacov and between Henson and Raynaud; also, Henson and Junge are in the same department. Nonetheless, the workshop provided an important opportunity for the four of them to discuss this new topic intensively together. In particular, the direct interaction between Ben Yaacov and Junge was fruitful and would likely not have occurred without the workshop.

Questions they are working on include:

- Axiomatisation of various classes of von Neumann algebras with a normalised trace.
- Quantifier elimination in the theory of the von Neumann algebra R.
- Notions of independence in non-commutative probability spaces. In particular, is the theory of R simple?
- Relations between various properties and constructions on groups (e.g., elementary theory and ultraproducts) and the analogous properties and constructions on the corresponding von Neumann algebras.

Nakano spaces.
Nakano spaces are a generalisation of L_p spaces in which p can vary. Henson's former student Poitevin studied the model theory of these spaces in his 2006 PhD thesis, and during the workshop his results were presented. Ben Yaacov solved some problems left open by Poitevin, and intends to pursue the matter further, alone or in collaboration with Poitevin.

Banach spaces.
Valuable interactions occurred within the Banach space work group, which was quite heterogeneous. From the start they faced the challenge of developing a common language for model theorists and functional analysts. The mathematical focus of the group evolved toward a problem that strongly involves both model theory and Banach space geometry. It seems likely that several members of this group will stay in contact around this problem and others like it. This will be especially valuable for the young researchers in the group.

Stability and simplicity of randomised structures.
Keisler presented his results concerning randomisations of first order structures as continuous structures. In particular, he proved that if M is $\aleph_0$-stable as a first order structure then its randomisation M^R is $\aleph_0$-stable as a continuous structure. Ben Yaacov gave an argument showing that if M is stable (or simple) then so is M^R. Ben Yaacov and Keisler agreed to continue working on this topic together towards common publication of the results and their further extensions.

Asymptotic cones.
Françoise Point is an established researcher and Sylvia Carlisle is a PhD student working with Henson on a thesis in this area. They established a good working relationship at the workshop and it has continued during the writing of their report. This contact will be helpful to Carlisle and may well lead to future collaborations between them.

Urysohn's metric space.
Julien Melleray started a postdoc in Urbana in August, 2006, but Henson is on sabbatical leave this semester in California, so the two of them had not met until the workshop. The Urysohn space was considered in the definable sets work group, and their discussions in this setting made it clear that each had much to teach the other about the model theory of this space. They have followed up long-distance on this beginning, and have begun a collaboration that will continue. It would very likely have begun eventually anyway, but the workshop started it sooner and with more focus than would otherwise have happened.

Further new activities.
Frank Wagner reports the work group on stable groups may resume activity, with some personnel changes. Alexei Kolesnikov reports he started working on some problems raised during the workshop with a graduate student at Michigan, Jessica Metcalf-Burton. José Iovino and Anand Pillay agreed to pursue issues around weak metric structures (see below).

New problems

Weak structures and theories.
To every continuous structure M one can associate a natural first order structure M^w, and correspondingly to every continuous theory T a first order theory T^w. These *"weak" structures and theories* can be used to reduce the Keisler-Shelah theorem for continuous logic to the classical first order version. A question which was posed and left unanswered was: assume T is $\aleph_0$-categorical. Is T^w complete?

Randomisations of first order theories T.
Dependent theories, also referred to by some as theories having the NIP (non independence property), form a class of particularly well-behaved theories, with properties similar to those of stable and simple theories. It was shown during the workshop that if T is stable (or simple) then so is its randomisation T^R. Pillay asked the following: Assume T is dependent. Is the randomisation T^R dependent as well?

Generalisations of Krivine-Maurey.
Krivine and Maurey proved an ℓ_p subspace theorem for Banach spaces that are stable for quantifier-free formulas in continuous logic. The Banach space work group discussed proofs of this result, and explored ways of carrying them out for more general classes of Banach spaces. In particular, they discussed how to prove the existence of ℓ_p-types without stability. Some ℓ_p subspace results are known for specific non-stable spaces, such as non-commutative L_p-spaces or weakly stable spaces (where c_0 also occurs), or using Maurey's so-called "strong ℓ_1-types. Thus the possibility of finding a unifying framework more general than stability seems worth pursuing.

Tsirelson's space.
During his presentation of this important Banach space (focusing on the 2-convexification of the dual of the space originally constructed by Tsirelson), Bill Johnson noted that every ultrapower of this space is linearly homeomorphic to a canonical direct sum of the space and a Hilbert space of suitable dimension. It should follow that every model of this continuous theory has the same form and hence the theory is uncountably categorical in the linearly homeomorphic sense (what Banach space theorists call the "isomorphic" sense). On the other hand, this Tsirelson space is not stable, not even in Krivine-Maurey's quantifier-free sense (or else it would have an ℓ_p subspace, which it famously does not have). This makes a more careful study of the model theory of this space very desirable, and several participants intend to carry this out (including at least Henson, Iovino, and Ortiz).

Definable sets.
This concerns the notion of definability for sets that is introduced in [BBHU, Section 9] During the workshop there was a lot of discussion about this notion and the associated terminology.

The terminological concern was that the word "definable" was already too central in model theory and potentially evocative of wrong ideas. After a fairly long discussion of alternatives (continued after the workshop ended) there emerged a somewhat reluctant consensus to use the phrase *distance definable* (or, for short, *d-definable*) and the authors of [BBHU] intend to revise it accordingly.

More interestingly, issues involving d-definability arose several times in the discussions of different work groups during the workshop. There is a clear need for this notion to be

studied further, in examples and in the general theory, and it is somewhat clearer after the workshop what one should try to do.

Here are some indications of how d-definability came up during the workshop (with more details to be given in several work group reports):

- Conjecture (Melleray): If $\mathbb{U}$ is the Urysohn metric space and φ an automorphism with small orbits (for example, if φ has finite order), then every non-empty d-definable set in $(\mathbb{U}, \varphi)$ is isometric to $\mathbb{U}$ itself.
- Carlisle and Point showed during the workshop that in a geodesic metric space (in particular, in the asymptotic cone of a finitely generated group), every closed ball is d-definable.
- Ben Yaacov introduced a notion of a theory having enough d-definable sets; this means that in any model, the set of realizations of any n-type is the intersection of a family of d-definable sets (over the same set of parameters). He gave an example of a continuous theory with no non-trivial d-definable sets, and asked whether every stable (or even $\aleph_0$-stable) theory has enough d-definable sets.
- The group working on non-commutative probabilities gave axioms for the class of II_1 factors relying heavily on the fact that quantification over d-definable sets is expressible in continuous logic.
- Henson gave an example of a pair of d-definable sets whose intersection is not d-definable.

Bibliography

[Ben] Itaï Ben Yaacov, *On perturbations of continuous structures*, submitted.

[BBHU] Itaï Ben Yaacov, Alexander Berenstein, C. Ward Henson, and Alexander Usvyatsov, *Model theory for metric structures*, submitted for publication in a Newton Institute volume in the Lecture Notes series of the London Math. Society, 109 pp.

[BU] Itaï Ben Yaacov and Alexander Usvyatsov, *Continuous first order logic and local stability*, submitted.