Conformal structure in geometry, analysis, and physics

organized by
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Workshop Summary

Significant progress was made by Robin Graham during and inspired by the meeting. He reported that, for conformally Einstein manifolds, the Fefferman-Graham ambient metric construction admits an essentially unique special formal solution to all orders. The result concerning uniqueness came during the workshop. This led him, during the meeting, to two further results concerning conformally Einstein structures. There is an important family of conformally invariant operators known as the GJMS (Graham, Jenne, Mason, and Sparling) operators. These have leading order a power of the Laplacian and so generalise the conformal Laplacian. In even dimensions they exist, for general conformal structures, only for even orders up to the dimension of the manifold. Graham proved that the new result for the ambient metric means that that on conformally Einstein manifolds, operators of all even orders exist. Graham also showed that, in this setting, these operators (in either dimension parity, and at an Einstein scale) can be expressed explicitly by a general formula as a composition of scalar curvature modified Laplacians. The GJMS operators, have a central role in conformal geometry; for example, they control conformal curvature variations for generalizations of the Yamabe problem (Q-curvatures of sub-dimensional homogeneity), and for the critical (dimension homogeneity) Q-curvature prescription problem. This latter problem is analogous in a PDE sense to the Gauss curvature prescription problem in dimension 2. Graham's simplification in the conformally Einstein setting should have deep consequences for geometric analysis. Inspired by Graham's result, Rod Gover observed that the results concerning factorization can also be recovered using tractor calculus, at least for those operators previously known to exist. Some of the results in the recent preliminary preprint, "On the renormalized volumes for conformally compact Einstein manifolds" of S.-Y. Alice Chang, Jie Qing and Paul C. Yang are an an outcome of discussions (and results heard) at the meeting. Tom Branson and Gover added the final ideas to the article "Conformally invariant operators, differential forms, cohomology and a generalisation of Q-curvature" (math.DG/0309085). This constructs a family of operators between differential forms that generalise the Q-curvature. These are used in several applications, including a conformally invariant de Rham Hodge theory and the construction of a new family of conformally invariant elliptic complexes involving differential forms. The latter complexes yield a new family of "half-torsions" which generalise Cheeger's half-torsion for the de Rham complex. Progress was also made in formulating this theory which generalises further to a give an analogous half torsion, or "detour torsion" (really a linear combination of functional log-determinants for a larger class of conformal generalized Bernstein-Gelfand-Gelfand complexes. Spyros Alexakis presented an argument for an invariant theoretic conjecture that has been central to the theory – roughly speaking, that any invariant that acts conformally "like an index" necessarily takes a certain form. One description of this form is as "a Q" (something with a linear conformal deformation law), plus an exact divergence. At this point, Alexakis is refining and writing up his argument, and there is general agreement that the statement is true, and the proof goes through using a careful nested induction which employs the fundamental ingredients that Alexakis presented. Still open is a possibly related conjecture about conformal primitives, which would lead to strong statements about higher-dimensional Polyakov formulas, expressing the difference of log-determinants (or of quantities like the generalized half-torsion mentioned above) at conformally related metrics.

Branson, Andreas Cap, Michael Eastwood, and Gover worked on an article entitled, "Prolongations of geometric over-determined systems". Though the results are not necessarily in conformal geometry, this provides one of the main motivations. Moreover, the methods arise from conformal differential geometric machinery recently developed by David Calderbank, Cap, Tammo Diemer, Jan Slovak, and Vladimir Soucek (all of whom, save for Diemer, participated in the AIM workshop). The results extend work of Uwe Semmelmann (presented at the workshop) and apply to twistor-spinor equations (as presented by Helga Baum). The final article will be posted on the preprint arXiv in the next few days. An important conformally invariant tensor quantity related to the Q-curvature and the ambient construction is the Fefferman-Graham ambient obstruction tensor. In fact, the workshop provided the impetus for Graham and Kengo Hirachi's final check of their announced result that the total metric variation of the Q-curvature is the obstruction tensor. The obstruction tensor generalises the Bach tensor of dimension 4, and is the obstruction to continuing the formal solution of the ambient metric problem in even dimensions. It was also shown to be a key ingredient in recent work of Gover with Hirachi which establishes the non-existence of GJMS type operators of order greater than the dimension on general even dimensional conformal manifolds. During the meeting Gover and Lawrence Peterson made progress on a program to give a new ambient description of these tensors, and describe them in terms of the tractor/Cartan calculus. There was also discussion among several participants about the computation of explicit spectra. A prototypical example of such a calculation is implied by the new result of Graham mentioned above. Though other derivations of the spectra of the GJMS operator at constant curvature metrics exist, some of the methods mentioned above, including Gover's tractor approach, seem promising for the computation of explicit spectra and spectral invariants in the future.