Complexity of mappings in CR geometry

organized by
John P. D'Angelo and Peter Ebenfelt

Workshop Summary

We begin our summary by recalling the three principal goals.

The first goal was to determine the fundamental notions of CR complexity and to prove sharp results about these notions. The second goal was to organize CR complexity theory into a broad framework that will be useful in CR geometry and also apply to other parts of mathematics. The third goal was to bring active senior researchers and young mathematicians together to work in a focused manner that will forge interactions and guide future research.

Twenty-seven mathematicians, including both senior and junior mathematicians, attended the workshop. Participants included four graduate students (Lebl, Lee, Tonejc, and Zhang) and four post docs (Chakrabarti, Herbig, Juhlin, and Peters). Most of these young people worked actively in small groups with the senior mathematicians present, and there is considerable evidence that the third goal was achieved.

The workshop had just two formal hour talks, held on Monday morning. In these talks each organizer described his own view of the developing subject of CR Complexity Theory. D'Angelo commented that while he could not define *CR Complexity Theory*, he knew it when he saw it. Both D'Angelo and Ebenfelt included definitions of complexity in various settings from CR Geometry. On Monday afternoon Salah Baouendi moderated a problem session. This problem session engaged the interest of most participants, and participants formed groups to work on these problems. The problem session reconvened twice during the week, and several *progress reports* were given.

The groups formed were fluid, and several people worked with more than one group. Three of the groups seemed to play the most significant roles. These groups were the *Finite Jet Determinators*, the *Proper Mappings between Balls* group, and the *Finite Mapping* group. These groups developed naturally from the opening talks and the first problem session, and they worked on specific problems. Two other groups were the *Positivity* group and the $\overline{\partial}$-group. The Positivity group interacted some with the Proper Mapping groups. The $\overline{\partial}$-group was somewhat negatively impacted by the absence of two invited people (Catlin, McNeal) who cancelled their trips for personal reasons. Senior mathematicians Fornaess and D'Angelo each worked with young mathematicians Lee and Kim on separate problems, and doing so helped them to be active participants.

The Proper Mapping group had considerable success. D'Angelo, Lebl, and Peters had expected, while at the workshop, to put the finishing touches on an essentially completed paper. On the final day, however, they made considerable improvements to the results in this paper, and hence it will be finished after the workshop. The main new result was an improved estimate for the degree of a proper monomial mapping between balls in terms of the domain and target dimensions. A simplified statement of their result is the estimate $d \leq \frac{4}{3}\frac{2N-3}{2n-3}$ for the degree d in terms of the domain dimension n and the target dimension N.

Furthermore these three people worked with Faran, Ji, Putinar, Meylan, and Zhang on related aspects of the complexity of proper mappings between balls. The group spent considerable effort on studying orthogonality issues arising in the rational case. Future progress is quite likely.

The main problem considered by the Proper Mapping group helps to explain what CR complexity might be. Given that $n \geq 2$ and $f : S^{2n-1} \to S^{2N-1}$ is a smooth CR function, then f must be the restriction to the sphere of a rational mapping. Can one estimate its degree in terms of the domain and target dimensions? The answer is yes, but obtaining a sharp estimate seems to be fundamental to formulating CR complexity as a subject. We can think of *degree* as a measure of complexity. It is natural to ask how complicated can a CR mapping between spheres be, given the domain and target dimensions, and what are the appropriate measures of complexity? Answers here will lead to formulating the basic notions of the field for general CR manifolds, as the unit sphere serves as a model for the strongly pseudoconvex case.

Thus the work of this group helped meet both the first and second goals of the conference and enlarged the number of mathematicians interested in the area. In particular Faran, who did important work on this problem in the early 1980's, has reconnected with these problems and is likely to contribute to them again.

The group working on finite jet determination addressed a conjecture by Baouendi–Ebenfelt–Rothschild: *Let M be a real-analytic holomorphically nondegenerate CR manifold and $p \in M$. Then there is an integer k such that any local analytic CR automorphism fixing p is uniquely determined by its k-jet at p.* Following an idea of Lamel and Mir, the group was led to try to find a partial normal form for holomorphically nondegenerate CR manifolds. The group made some progress on this problem during the workshop. A partial normal form was constructed for certain classes of such manifolds. It was noted that a particularly tricky case would arise if the CR dimension of such a manifold is larger than 3 and the Levi form has constant rank 1 everywhere. On the other hand, it was also noted that no examples of such manifolds are known in the literature. During the workshop, the group was able to construct a class of such examples (e.g. the tube over the second osculating surface of a sufficiently generic curve in $\mathbf{R}^4$). A PDE approach to a more systematic study of these manifolds was also proposed and will be pursued after the workshop.

The Finite Mapping group considered a problem posed by Rothschild in the problem session. Suppose that $f : (\mathbf{C}^n, 0) \to (\mathbf{C}^n, 0)$ is a finite map germ, and that V is a subvariety of the target space. If the inverse image $f^{-1}(V)$ is smooth at 0, must V then be smooth at 0? This problem generated considerable discussion in the problem session. It was observed that the result is false over fields of finite characteristic. It is known to hold when V has dimension 1. The group neither proved the result nor found a counterexample, but made some progress in the hypersurface case. A subgroup (including at least Stanton and Whittlesey) decided to pursue this question in a collaborative effort after the workshop.

The Positivity group worked on a problem posed by D'Angelo. Suppose that M is the strongly pseudoconvex boundary of a bounded domain, that M is defined by a polynomial equation, and that R is a Hermitian symmetric polynomial that is positive on M. Are there finitely many *holomorphic* polynomials f_j such that $R = \sum |f_j|^2$ on M. This result is known for the unit sphere, and for the sphere it is closely related to the work of the Proper Mapping group. Similar sounding results hold for more general domains, but these results do not provide a key piece of information: the f_j are polynomials when R is. The positivity group included Putinar, whose expertise in similar questions enabled him to pose a fascinating question about extending a famous algorithm of Schur to several complex variables. Schur's algorithm nicely connects geometric ideas such as Schwarz's Lemma with positivity conditions, and the connections forged between CR Geometry and Functional Analysis by this discussion seem promising. These developments give further evidence of success in meeting the third goal.

Kim worked on a problem posed by D'Angelo. Let Γ be a finite subgroup of the unitary group, and consider a group-invariant CR mapping from a sphere to a (minimal) hyperquadric. How are the numbers of positive and negative eigenvalues in the defining form for the target hyperquadric related to the representation of the group? Kim showed, for cyclic groups (asymptotically as the order tends to infinity) that the fraction of positive eigenvalues tends to $\frac{3}{4}$, whereas for dihedral groups it tends to $\frac{1}{2}$. This work suggests connecting CR complexity to representation theory and algebra.

There is no doubt that progress on several problems in CR Geometry occurred during the workshop. Several specific new results were proved, and promising new collaborations were formed. Senior and junior mathematicians interacted well. The developing subject of CR complexity theory has advanced because of this workshop. Part of the first goal was to determine the fundamental notions of CR complexity. Progress at the workshop shows that, while the fundamentals of the subject were not completely worked out, an even better thing took place. Participants studied and made progress on diverse problems all fitting into a broad framework of Complexity Theory in CR Geometry.