Cohomological methods in abelian varieties

organized by
Alexander Polishchuk, Alexei Skorobogatov, and Yuri Zarhin

Workshop Summary

This workshop focused on cohomological invariants of abelian varieties, in particular, those with integral coefficients and Brauer groups, as well as invariants used in the study of rational points, such as Selmer groups and Shafarevich–Tate groups. Our purpose was to bring together researchers with quite different perspectives: from classical algebraic geometers and experts in motives to people doing arithmetic algebraic geometry and number theory. We feel that the workshop was very successful in promoting interactions between these different groups of mathematicians.

The workshop had the AIM format of two lectures in the morning and discussions and working groups in the afternoon.

One of the main topics of the workshop was the study of Mordell–Weil, Shafarevich–Tate and Selmer groups of elliptic curves and abelian varieties. It was the subject of two lectures of Karl Rubin and lectures of Douglas Ulmer and Bjorn Poonen. (Ulmer also led a discussion group devoted to a remarkably elementary construction of curves of given genus over the function field $\mathbf{F}_p(t)$ of characteristic p with explicit points of their Jacobian generating a group of arbitrarily large rank.) Rubin explained how a natural generalization of Selmer groups leads to bounds for Shafarevich–Tate groups. In his second lecture he studied the distribution of 2-Selmer ranks in the family of quadratic twists of an arbitrary elliptic curve over an arbitrary number field. Ulmer outlined a general approach to a construction of curves over $\mathbf{F}_p(t)$, whose Jacobians have unbounded ranks in the tower of fields $\mathbf{F}_p(t^{1/d})$ while d tends to infinity. Poonen discussed an approach to the study the distribution of p-ranks of Selmer groups over a given global fields that is based on a natural realization of Selmer p-groups as random maximal isotropic subgroups in certain locally compact $\mathbf{F}_p$-vector spaces.

One of the directions of the workshop was to discuss integral motives of abelian varieties. We had an introductory talk by Ben Moonen to provide some background for studying this question which was followed by the questions and answers session in the afternoon. The working group on integral motives of an abelian variety considered the problem of existence of an integral version of the Deninger–Murre decomposition and the related question of whether the Hochschild–Serre spectral sequence for the étale cohomology of A/k with finite or integral coefficients degenerates. With regard to the last question we followed the suggestion of Skorobogatov to look at the Jacobian J of the generic fiber of the universal family over the moduli space of curves of genus 2. In this case the nontriviality of the differential on the second page would follow from the non-divisibility by 2 of the Galois cohomology class in $H^1(G_k, J)$ obtained from the torsor of theta-characteristics. This cohomology class is obtained from a similar class defined for the fundamental group of the entire moduli space $\mathcal{M}_2$ which seems to be computable and is related to the torsor of quadratic forms on $(\mathbb{Z}/2\mathbb{Z})^4$

inducing the standard symplectic form. In a different direction, this working group considered the question of extending the definition of symmetric powers to integral motives. The related topic of the degeneration of Leray spectral sequence for a family of abelian varieties (for cohomology with integral coefficients) was addressed in a talk by Claire Voisin. Claire also led a discussion group devoted to the question of how one can use infinitesimal invariants to prove that certain cycles on Jacobians of generic curves are not algebraically equivalent to zero. After reviewing the classical construction we considered its potential application to Jacobians. The main difficulty is studying the relevant infinitesimal invariants for generic curves. In a particular case, the computational problems are similar to those appearing in the Green syzygy conjecture (resolved by Voisin).

A working group led by Yuri Zarhin concentrated on the Tate conjectures for divisors and their versions with finite coefficients, with application to the finiteness conjecture of the Galois invariant part of the geometric Brauer group (proved by Zarhin and Skorobogatov for abelian varieties and K3 surfaces). It seems plausible that these results could be extended to some other classes of varieties.

A group led by Bjorn Poonen worked on efficient calculation of the Picard group of algebraic varieties over non-closed fields. Bruno Kahn led another working group that discussed the generalized Hodge and the generalized Tate conjectures for products of CM elliptic curves. Ben Moonen led a discussion group on abelian varieties of Weil type.

Yuri Zarhin and Rachel Pries discussed endomorphisms ring of Jacobians.

We believe that our main objective has been reached. We confidently expect progress on most of the problems discussed at the main lectures and the numerous working groups, and expect some of these to be solved in the near future.