The Cohen-Lenstra heuristics for class groups

organized by
Manjul Bhargava, Juergen Klueners, Joe Buhler, and Benedict Gross

Workshop Summary

An AIM workshop on "The Cohen-Lenstra heuristics for class groups" was held from June 13–17, 2011. There were approximately 35 participants, including both Cohen and Lenstra, as well as many other participants from Asia, Europe, and North America. A number of mathematicians in the local Palo Alto area also participated.

The aim of the conference was to discuss several questions arising from the various exciting recent developments surrounding the Cohen-Lenstra heuristics. The Cohen-Lenstra heuristics were formulated about 30 years ago and have since become an important guiding principle in understanding the distributions of arithmetic objects. However, little progress had been made on these heuristics for perhaps the first 20 years. It is only quite recently that many strides have been made: first, in proving some special cases of these conjectures; second, in locating counterexamples to the conjectures due to "small primes"; and finally, extending and correcting these conjectures in many natural contexts. Moreover, these lines of progress have come from many different areas within mathematics, including analysis, probability, and geometry, and of course analytic and algebraic number theory.

The conference brought together people from all these diverse areas, so that cross-disciplinary interaction relating to the problem could take place. Each morning there were a couple of introductory lectures on the various aspects of the Cohen-Lenstra heuristics described above. The lectures were also chosen to provide a good source of discussion as well as problems to work on during the afternoon sessions.

The first morning was inaugurated with a lecture by Poonen, who gave a clear exposition of the statements and reasoning behind the Cohen-Lenstra heuristics. Lenstra then followed with a lecture on some new results and problems relating to mass formulas for algebraic structures, a theme which was continued in a lecture by de Smit later in the week. The second day began with a motivational lecture by Ellenberg giving an overview about recent progress in the function field case. Bhargava followed with a lecture on some natural variations of the Cohen-Lenstra heuristics, to orders, ray class groups, and non-abelian settings, and some results in these directions. On the third day, there was a short talk by Yu on a remarkable computation in the hyperelliptic setting, followed by another short talk by Garton giving a nice theoretical explanation for Malle's modifications of the Cohen-Lenstra heuristics when roots of unity are present in the base field. Finally, there was a talk by Delaunay on his recent Cohen-Lenstra-style heuristics for the Tate-Shafarevich groups of elliptic curves.

On Thursday, Poonen spoke on his new heuristics with Rains on the distribution of Selmer groups of elliptic curves. Ho followed with a lecture describing some recent results using geometry-of-numbers that prove some cases of these Selmer group heuristics. On Friday, we had a talk from a more probabilistic viewpoint by Maples, who explained that the Cohen-Lenstra distribution on finite abelian p-groups is the same as the distribution one obtains by considering cokernels of random p-adic matrices, and indeed, this fact remains true in a way that is essentially independent (aside from degenerate cases) of the probability measure on this set of matrices. A random walk interpretation was also discussed. This was followed by a talk by Fouvry, who explained the recent proof with Klüners of the Cohen-Lenstra-Gerth heuristics on the 4-rank of the class group of quadratic fields. Finally, there was a nice short talk by Rubinstein-Salzado on the computation of certain invariants via which one can hope to understand the discrepancies arising in Cohen-Lenstra situations involving roots of unity.

The afternoon sessions were dedicated to the discussion of problems; these problems involved obtaining: 1) a satisfactory conjecture for the distributions of unramified nonabelian extensions of quadratic fields; 2) an Iwasawa perspective on the Cohen-Lenstra heuristics; 3) analogues of these heuristics for the Arakelov class group; 4) an understanding of the Cohen-Martinet extensions of the Cohen-Lenstra heuristics to non-Galois and higher degree fields; 5) a deeper understanding of Malle's extensions of the Cohen-Lenstra heuristics for roots of unity; 6) an answer to a question of de Smit on mass formulas; and 7) a better understanding of statistical and computational issues surrounding Cohen-Lenstra.

Some of these problems were worked on very hard by many people and survived until the end of the week, and due to initial or significant progress, participants are still thinking about them. This is the case (at least) for 1), 5), 6) and 7) above. In the case of 7), we had the pleasure one afternoon of having a discussion session with Diaconis which was reported to be of great help. On 2) and 3) not as much progress was reported, but interest remained. For example, it was suggested to ask an Iwasawa theory expert after the conference; the additional idea arose that the basic constructs of Iwasawa theory too ought to be looked at from the heuristic/statistical point of view.

During the conference dinner, there were insightful and entertaining reminiscences by Cohen and Lenstra on their original discovery of these heuristics. Overall, the conference was reported to be a wonderful learning experience by many, as well as an opportunity to interact and connect with mathematicians (not necessarily in their own field) having a common interest in random finite abelian groups and related structures arising in arithmetic and beyond.

Last but not least, serious consideration is being given to publishing a volume containing papers on the subject, many drawn from the lectures and discussions at the workshop. Details to follow...