Brownian motion and random matrices

organized by
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Workshop Summary

This workshop, sponsored by AIM and NSF, was devoted to β-generalizations of the classical ensembles in random matrix theory. Recent advances have put stochastic methods on center stage, thus explaining the workshop title 'Brownian motion and random matrices'.

One recalls that a viewpoint on classical random matrix theory, generalizing Dyson's three fold way, is that physically relevant ensembles are specified by the Hermitian part of the ten infinite families of matrix Lie algebras. By specifying a Gaussian weight, in each case the corresponding eigenvalue probability density function can be identified with the Boltzmann factor of a classical gas interacting via a repulsive pairwise logarithmic potential (log-gas). Curiously, the dimensionless inverse temperature β in this analogy is restricted to one of three values $\beta = 1$,2 or 4. This analogy, with the same restriction on β, carries over to the classical ensembles of random unitary matrices based on the ten families of symmetric spaces in correspondence with the matrix Lie algebras.

In work dating from the first half of the previous decade, explicit constructions were given of random matrix ensembles with eigenvalue probability density functions realizing the log-gas Boltzmann factors for general $\beta > 0$. In the cases of the classical Gaussian Hermitian and circular ensembles, these construction are in terms of certain tridiagonal and unitary Hessenberg matrices respectively. Alternatively it was shown that the β-ensembles could be realized by certain families of random matrices defined recursively.

In the second half of the the previous decade it was shown that the tri- (and bi-) diagonal matrices appearing in the construction could be viewed as discretizations of certain differential operators perturbed by a noise term involving Brownian motion. Similarly, by analyzing the recurrences satisfied by the characteristic polynomials associated with the tridiagonal and unitary Hessenberg matrices, stochastic differential matrices were derived for the charaterization of the number of eigenvalues in a given interval in the bulk. This in turn was used to solve some previously intractable problems in random matrix theory, an example being the large distance asymptotic expansion of the spacing distributions for general β.

The AIM workshop 'Brownian motion and random matrices' sought to build on these advances, and to tackle other problems of fundamental importance to random matrix theory. Three classes of problems were so identified.

- Universality. Do the bulk scaled eigenvalues in the β-generalized Gaussian and circular ensembles have the same distribution, and what if the Gaussian is replaced by say a quartic? Seemingly different stochastic descriptions apply in these cases, and the task is to show that they are in fact identical.

- Phase transitions. Gaussian ensembles can be generalized to have Brownian motion valued entries, with one of the simplest initial conditions being to start all eigenvalues off at the origin except for one outlier. By tuning the value of the position of the outlier as a function of N, it is possible to get a critical regime, which is essentially the one studied in the context of spiked models. The problem here is to use the tridiagonal matrix models to study this setting, and to apply the findings to spiked models.
- Integrability. Random matrix theory is a rich arena of integrability, with key probabilistic quantites known in terms of solutions of certain (non-stochastic) d.e.'s. One would like to use the s.d.e.'s, or other structures not restricted to the classical couplings, as a pathway to exact results for general β.

The workshop began with talks by two of the three conference organizers: Forrester and Virág. The talk of Forrester was structured about these three classes of problems, while the talk of Virág surveyed the developments stemming from the introduction of stochastic differential equations and related concepts into random matrix theory. Time was then spent on identifying specific problems for research. In this process the initial list was further specialized, and problems outside the list were formulated. An example of the latter was to quantify the random function implied by the bulk scaling limit of the characteristic polynomial for the β-ensembles. Four of these problems were chosen for future research during the week, and substantial progress was made on two: recurrences relating to alternative constructions of the β-ensembles, and spiked models at the hard edge.

In relation to the first, one recalls that the β-ensembles were originally formulatted in terms of tri-diagonal and Hessenberg matrices, and it was from these that the characterizations of β-ensemble states in terms of stochastic differential equations were derived. An alternative formulation of the β-ensembles is via Selberg integral theory. The problem posed was to investigate the scaling limit of the corresponding recurrences for the characteristic polynomials, which are distinct from those implied by the original constructions. These recurrences are in fact related to certain generalized eigenvalue problems, although no use was made of this feature. Instead a more direct analysis was employed. In the case of the Laguerre ensemble, the recurrence was shown to remain a recurrence in the hard edge scaling limit; for the circular Jacobi ensemble two regimes were identified — one about the spectrum singularity and the other about the point furtherest from the spectrum singularity. The latter both appear to relate to stochastic differential equations in the scaling limit. Moreover, in the course of this investigation it was noted that the spectrum singularity should be regarded as a Fisher-Hartwig singularity. The Fisher-Hartwig singularity has two parameters — one the exponent for the vanishing at the spectrum singularity, and the other controlling a discontinuity. A basic observation (due to Rains) made at the workshop is that by taking a limit of the latter the hard edge is reclaimed, so there is a transition from the two-parameter spectrum singularity to the hard. This fact seems to have escaped earlier attention.

In the original list of problems, it was proposed to use tridiagonal models and the methods of stochastic differential equations to study the scaling limit of the separation of the largest eigenvalue under a low rank perturbation. It turned out that just before the workshop Bloemendal and Virág had made substantial progress on the analogous effect for spiked Wihart matrices. This motivated the question of using these methods to study separation of the smallest eigenvalue in spiked Wishart models. This is a hard edge effect and thus complementary to the soft edge effect studied by Bloemendal and Virág.

The insight of Bloemendal-Virág was that critical spiking (in the population matrix) transformed the boundary condition in the charactering random operator from Dirichlet to "mixed". In the hard edge case, the random operator generates a diffusion process and the mixed boundary condition has a separate probabilistic meaning. Again one can write down PDE's for the distribution functions of now the spiked minimal Wishart eigenvalues. The consequences of this were considered at the workshop by Baik, Ramirez, Rider, Sutton, and Zeitouni.

During the week a further working group was formed around the topic of the random analytic function formed from the characteristic polynomial of the β ensembles in the bulk. Virág and coworkers have used the stochastic description to quantify this entity for $\beta > 2$. Attention was focussed on the case $\beta = 2$, and the approach taken was to seek to analyze the moments.

The two talks that began the workshop have already been recorded. On each subsequent day a number of other talks were given. In particular, Killip spoke on an approach to the averages of multiple characteristic polynomials in the circular β-ensembles using partial differential equations; Bloemendal on low rank perturbations of Wishart β-ensembles and their description in terms of stochastic differential equations; Borodin on determinantal point processes relating to the KPZ equation of stochastic growth; Valkó and Kritchevski on localization in a discretization of the one-dimensional Schrödinger equation in the presence of noise; Rains on the use of Selberg integral theory to construct β-ensemble; Breuer on the spectral measure associated with the β-ensembles; Krishnapur on random matrix and random function models with complex eigenvalues and zeros; and AIM director Conrey spoke on random matrix theory as it occurs in number theory.