Brauer groups and obstruction problems: moduli spaces and arithmetic

organized by
Asher Auel, Brendan Hassett, Anthony V'arilly-Alvarado, and Bianca Viray

Workshop Summary

This workshop brought together experts from two different fields: number theorists interested in rational points, and complex algebraic geometers working on derived categories of coherent sheaves. We were motivated by fresh developments in the arithmetic of K3 surfaces, which suggest that cohomological obstructions to the existence and distribution of rational points on K3 surfaces can be fruitfully studied via moduli spaces of twisted sheaves. In turn, the birational geometry of these moduli spaces has been the object of recent study via the theory of stability conditions.

Our aim was to encourage cross-pollination between the two fields and to explore concrete instances when the derived category of coherent sheaves on a variety over a number field determines some of its arithmetic. For example, during the workshop the following question was raised and investigated: Given smooth projective varieties X and Y over a number field k such that $D^b(X) \cong D^b(Y)$, must X and Y have the same places of bad reduction?

As per the usual AIM format, we had two talks every morning, with a long break in between. Early on in the week, the talks were mostly expository, emphasizing the connections drawn between the two fields through the concepts of "Hodge isogeny" and "Brauer classes as obstructions to fineness of moduli spaces", as explained through the examples of descent on elliptic curves, the Tate-Shafarevich group of an elliptic fibration, cubic fourfolds containing a plane, and K3 surfaces of degree 2. There were also expository talks on the Brauer–Manin obstruction to the Hasse principle for the existence of rational points and on the derived categorical interpretation of Hodge isometry and isogeny between complex K3 surfaces.

On Monday afternoon, Danny Krashen moderated a problem session; the activity lasted all afternoon. We divided the problems thematically into six groups, which formed the basis for the group discussions on Tuesday, Wednesday, and Thursday. On Friday afternoon, representatives from each of the groups that met at least once presented a summary of the group's findings. These presentations are synthesized below.

We hope that the workshop was only the beginning of many worthwhile collaborations and future lines of research. To this end, we are preparing a proceedings volume documenting some of the background for the workshop, as well as the results obtained by working groups during and after the workshop.

Summary of working groups

Determining the Galois action on $Br(\overline{X})[2]$ for a degree 2 K3 surface X..
Presenter: Alexei Skorobogatov

Let $C \subset \mathbb{P}^2$ be a smooth sextic curve and let X be the double cover of $\mathbb{P}^2$ branched over C. The aim of the group was to compute $\mathrm{Br}(\overline{X})[2]$ as a Galois module. They showed that if $\text{Pic}(\overline{X})$ was rank 2 and generated by the hyperplane class H and a divisor D such that $D.H$ was odd, then $\text{Br}(\overline{X})[2] \cong \text{Jac}(C)[2]$, *as Galois modules.* In general, they showed that a quotient of $\text{Jac}(C)[2]$ was isomorphic to a subgroup of $\text{Br}(\overline{X})[2]$ of index at most 2. The group plans to continue working to determine the Galois structure of the full group of 2-torsion Brauer classes.

Good reduction and derived equivalence.
Presenter: Max Lieblich

Let X, Y be varieties over a number field k. The group considered whether the places of bad reduction are a derived invariant, i.e., if $D^b(X) \cong D^b(Y)$, then do X and Y have the same places of bad reduction? If X and Y are elliptic curves and $D^b(X) \cong D^b(Y)$, then $X \cong Y$, so they obviously have the same places of bad reduction. An alternate proof, which can be generalized, is that if X and Y are derived equivalent elliptic curves, then $H^1(\overline{X}, \mathbb{Z}_\ell) \cong H^1(\overline{Y}, \mathbb{Z}_\ell)$ as Galois modules. Combined with the work of Néron, Ogg, and Shafarevich, one concludes that X and Y have the same places of bad reduction. The group also showed that the question has a positive answer in the case of genus 1 curves. When X and Y are K3 surfaces, the group believes that there is a positive answer, and is planning to continue to work out the details.

Isogenies.
Presenter: Max Lieblich

For K3 surfaces X and Y, a result of Mukai and Orlov is that $D^b(X) \cong D^b(Y)$ is equivalent to a Hodge isometry $T(X) \cong T(Y)$. In the twisted case, a result of Căldăraru is that if $D^b(X, \alpha) \cong D(Y)$ then $T_\alpha(X) \cong T(Y)$, where $T_\alpha(X) = \ker(\alpha : T(X) \to \mathbb{Q}/\mathbb{Z})$. This group considered Hodge isogenies $\Lambda \subseteq T(X)$ such that $T(X)/\Lambda$ is *not* cyclic?

Question: For gerbes $\mathscr{X} \to X$ and $\mathscr{Y} \to Y$. Is there a Hodge theoretic criterion for $D^b(\mathscr{X}) \cong D^b(\mathscr{Y})$?

Conjecture: $D^b(\mathscr{X}) \cong D^b(\mathscr{Y})$ if and only if there is a Hodge isometry of "hyper-Mukai" lattices $HM(X, \alpha, \mathbb{Z}) \cong HM(Y, \beta, \mathbb{Z})$.

Derived equivalence of genus 1 *curves.*
Presenter: Ben Antieau

Let X, Y be derived equivalent genus 1 curves over k. If X or Y has a k point, then it is known that $X \cong Y$. Even if $X(k) = Y(k) = \emptyset$, it's not too difficult to prove that $\text{Jac}(X) \cong J(Y)$. This group has the following result, valid for any abelian variety A over a field k.

Let $X, Y \in H^1(k, A)$. Assume that $\text{char}(k) = 0$, $\text{End}(\overline{A}) = \mathbb{Z}$, and $\text{rank}(\text{NS}(\overline{A})) = 1$. Then $D^b(X) \cong D^b(Y)$ if and only if X and Y generate the same subgroup of $H^1(k, A)$.

The proof utilizes the recent theory of twisted forms of derived categories.

Geometric constructions.
Presenter: Justin Sawon

Aim: Describe a 3-torsion Brauer class on a degree 2 K3 surface.

As known to Mukai, starting with a degree $2n^2$ K3 X, one gets a degree 2 K3 surface $S = M_X(n, h, n)$ as a coarse moduli space of sheaves on X, the obstruction to fineness being a class $\alpha \in \mathrm{Br}(S)[n]$.

First, the group explained the geometry behind this in the case $n = 2$, in such a way that is amenable to generalizing to the case $n = 3$. This description is as follows. When $n = 2$ one starts with a degree 8 K3 surface X given as a complete intersection of three quadrics in $\mathbb{P}^5$; we embed $\mathbb{P}^5$ by the 2-uple embedding in $\mathbb{P}^{20}$. Taking projective duals, $\hat{\mathbb{P}}^5$ intersected with the plane $\mathbb{P}^2$ generated by the three quadrics, is a plane sextic curve $C \subset \mathbb{P}^2$. Then $S = M_X(2, h, 2)$ is the double cover of $\mathbb{P}^2$ branched over C. For each point $p \in \mathbb{P}^2$, we get a quadric fourfold $Z = \mathrm{Gr}(2, 4) \subset \mathbb{P}^5$ which comes with tautological sub and quotient rank 2 bundles E, F. Then $E^\vee|_X$ and $F|_X$ are stable bundles on X and have the desired invariants. Let W be the relative variety over $\mathbb{P}^2$ of planes in the quadric fourfolds Z; the Stein factorization of the map $W \to \mathbb{P}^2$ recovers the associated 2-torsion Brauer class of degree 2 on S.

Now consider $n = 3$. Let X be a degree 18 K3 surface; it embeds into a 5 dimensional projective homogeneous variety $Y := G_2/P$, which itself embeds into $\mathbb{P}^{13}$ via the adjoint representation of G_2. Taking projective duals, $\hat{Y} \hookrightarrow \hat{\mathbb{P}}^{13}$, intersected with a plane $\mathbb{P}^2$, again gives a plane sextic curve $C \subset \mathbb{P}^2$. Then $S = M_X(3, h, 3)$ is the double cover of $\mathbb{P}^2$ branched over C. Each point $p \in \mathbb{P}^2$ corresponds to a hyperplane $H_p \subset \mathbb{P}^{13}$, and the intersection with Y gives a Fano 4-fold Z which is degree 5 and index 2. Each such Z embeds uniquely in $\mathrm{Gr}(3, 6)$ and we proceed as in the $n = 2$ case to obtain a pair of stable sheaves on X. Letting W be the relative variety over $\mathbb{P}^2$ of cubic surface scrolls contained in the Fano 4-folds Z, the Stein factorization again recovers the associated 3-torsion Brauer class on S.

Bounding $Br_1(X)/Br(k)$.
Presenter: Andrew Obus

On Tuesday afternoon, one group discussed whether there could be an absolute bound of $\#\big(\mathrm{Br}(X)/\mathrm{Br}(k)\big)$, for any K3 surface X of fixed degree, and k a fixed number field. That question proved very difficult, but the group was able to answer a similar question, concerning $\#\big(\mathrm{Br}_1(X)/\mathrm{Br}(k)\big)$. Precisely, they proved: There exists a positive integer $N = N(n)$ such that for any field k and any variety X whose Picard group is free and rank at most n, then $\#H^1(k, \mathrm{Pic}\overline{X})|N$. The proof relies on the fact that there are only finitely many isomorphism types of finite groups embedded in GL_n, and that if G is a finite group, then $\#H^1(G, \mathbb{Z}^n)$ is absolutely bounded, regardless of the action.

Unramified cohomology of cubic fourfolds.
Presenters: Jean-Louis Colliot-Thélène and R. Parimala

Let X be a general cubic fourfold over $\mathbb{C}$ containing a plane. By projecting from the plane, we obtain a quadric surface bundle $Y \to \mathbb{P}^2$ with degeneracy locus $C \subset \mathbb{P}^2$ a smooth sextic.

Motivating question (a major open problem in algebraic geometry): Is X birational to $\mathbb{P}^4$?

If X is birational to $\mathbb{P}^4$ over $\mathbb{C}$, then for all fields $F \supset \mathbb{C}$, X_F is birational (over F) to $\mathbb{P}^4_F$. In particular, the Chow group $A_0(X_F)$ of 0-cycles of degree 0 should be trivial, and the unramified cohomology group $H^3_{\mathrm{ur}}(F(X)/F, \mathbb{Z}/2\mathbb{Z})$ should be reduced to the scalars $H^3(F, \mathbb{Z}/2\mathbb{Z})$. (Note that the Brauer group of a cubic fourfold is trivial over any field.) The aim is to either prove these directly (thereby showing that the motivating question is very hard) or disprove these (thereby proving the non rationality of the generic cubic fourfold containing a plane). The group made no progress concerning $A_0(X_F)$. However, concerning H^3_{ur}, they completely solved the problem.

Let X be a general cubic fourfold containing a plane over $\mathbb{C}$. Then $H^3_{\mathrm{ur}}(F(X)/F, \mathbb{Z}/2\mathbb{Z}) = H^3(F, \mathbb{Z}/2\mathbb{Z})$ for every field $F/\mathbb{C}$.

The proof uses, as a starting place, the work of Arason, Kahn, Rost, and Sujatha, that for a general quadric surface Q over a field k, the natural map

$$H(k, \mathbb{Q}_2/\mathbb{Z}_2(2)) \to H^3_{\mathrm{ur}}(k(Q)/k, \mathbb{Q}_2/\mathbb{Z}_2(2))$$

is surjective and the kernel is described in terms of certain 3-Pfister forms. The idea is then to push a class in $H^3_{\mathrm{ur}}(F(X)/X_F, \mathbb{Z}/2\mathbb{Z})$ to the generic fiber of the quadric surface bundle $Y \to \mathbb{P}^2$, yielding a class in $H^3_{\mathrm{ur}}(F(Y)/F(\mathbb{P}^2), \mathbb{Z}/2\mathbb{Z})$, then represent by a class ξ in $H^3(F(\mathbb{P}^2), \mathbb{Q}_2/\mathbb{Z}_2(2))$. Wishing to prove that ξ is unramified, one computes residues, finding that they could only be nontrivial along C or any other "bad" sextic curve tangent to C (this uses the genericity hypothesis and the geometry of the K3 surface $S \to \mathbb{P}^2$, which has Picard rank 1 by hypothesis). Then one subtracts away the class of $\alpha \cup (f)$, where α is the Brauer class of an appropriate model of the Clifford algebra associated to $Y \to \mathbb{P}^2$, and f is a parameter at each of the possible "bad" curves. One finds that $\xi - \alpha \cup (f)$ is unramified. Smartly choosing f to be a local norm from $S \to \mathbb{P}^2$, then a Pfister form computation finally shows that $\alpha \cup (f)$ vanishes in $H^3_{\mathrm{ur}}(F(Y)/F(\mathbb{P}^2), \mathbb{Z}/2\mathbb{Z})$, proving the result.