Braid Groups, Clusters and Free Probability

organized by
Jon McCammond, Alexandru Nica, and Victor Reiner

Workshop Summary

Organizers' report

The meeting and its goals

During the week January 10-14, 2005, the American Institute of Mathematics, in Palo Alto, California held a workshop on the emerging mysterious numerical coincidences that involve

- the dual Garside structures on the braid groups (and other Artin groups of finite type),
- the cluster algebras of Fomin and Zelevinsky of finite type, and the corresponding generalized associahedra,
- the theory of free probability and its close connection with the lattice of noncrossing parititions, and
- the ad-nilpotent ideals in the Borel subalgebra of a semisimple Lie algebra as encoded in the poset of nonnesting partitions.

The workshop, entitled "Braid groups, clusters and free probability", was organized by Jon McCammond, Alexandru Nica, and Victor Reiner.

Prior to the meeting itself, the main connections tying these topics together were mostly numerological. Similar numbers were showing up in each of the areas, various bijections had been conjectured but essentially none of them had been proven. There was a definite feeling that there *should* be some actual connections, but nothing concrete had appeared.

An equally important goal for the meeting was to educate the researchers from the various groups in each others' viewpoints, to facilitate cross-fertilization.

Meeting the goals

The first two days of the conference were devoted to general talks, generally of about 2 hours in length, divided by a break, and with the lecturers encouraged to give "homework" problems during the talk for people to work on and calculate at their seats. These talks were:

(1) Vic Reiner- the (type A) noncrossing partitions, nonnesting partitions, and associahedra
(2) Andu Nica - noncommutative probability theory and noncrossing partitions
(3) Jon McCammond - Garside structures, braid groups and noncrossing partitions
(4) Patric Dehornoy - Garside structures
(5) Sergey Fomin - cluster algebras of finite type and generalized associahedra
(6) Eric Sommers - ad-nilpotent ideals, the root poset and antichains of positive roots

The remaining days of the conference were devoted to problem sessions and talks given by various participants at the request of the organizers and other participants. A detailed outline of the problem-sessions was written by Drew Armstrong, and can be found at the AIM web-site at

`www.aimath.org/WWN/braidgroups`.

Progress made/announced

Connecting noncrossing partitions and generalized associahedra.

During the course of the week two announcements were made which dramatically improved our understanding of this connection.

First, Tom Brady announced (in a joint work with Colum Watt) that there was a natural geometric model for the lattice of noncrossing partitions which turns out to be the positive part of the generalized associahedron. As a consequence of this construction there is now a uniform proof that the noncrossing partition lattices really are lattices (a goal that has eluded researchers for the past several years) in addition to providing a way to mediate the connection between noncrossing partitions and generalized associahedra.

Second, Nathan Reading announced a bijection between noncrossing partitions and the vertices of generalized associahedra (clusters) which passes through an intermediate set of newly-defined objects: the *Coxeter-sortable* elements of the Coxeter group W.

Complex reflection groups?.

David Bessis explained his very recent preprint on noncrossing partitions for *well-generated* complex reflection groups (those which can be generated by n reflections when acting in $\mathbf{C}^n$), along their connection to Garside structures and the $K(\pi, 1)$ problem for Artin groups associated to these complex reflection groups.

This played in important role in clarifying how far into the world of complex reflection groups one might expect to generalize noncrossing partitions; this question had been raised by his earlier work with Ruth Corran on a particular infinite family of complex reflection groups.

Extended generalized associahedra.

Sergey Fomin and Nathan Reading circulated a preprint on their recent work, defining "extended versions" of generalized associahedra, which include as special cases the type A and B versions studied indepndently by Christos Athanasiadis and Eleni Tzanaki.

This completed a circle of "extended" versions already known for the noncrossing partition lattices (defined and studied by Drew Armstrong) and for the antichains of positive roots (defined and studied by Athanasiadis).

Collaborations initiated

The AIM conference played a role in these developments in that it brought the key participants face-to-face, often for the first time, it focused everyone's attention on producing results in time for this unprecedented meeting, it enabled participants to learn results from the other areas from the experts in the field, everything from the basic introductory results to

the latest announcements, and since the conference has ended, it has facilitated the creation of a number of research articles.

For example, Christos Athanasiadis, Tom Brady, and Colum Watt produced a preprint (math.CO/0503007, *Shellability of noncrossing partition lattices*), resolving a conjecture of Reiner.

David Bessis and Vic Reiner are working on explaining an instance of the *cyclic sieving phenomenon* which seems to show up in the noncrossing partition lattices.

Website

A semi-permanent website has been set up at

`www.math.ucsb.edu/ ∼ jon.mccammond/associahedra`

for the researchers working in this emerging confluence of areas, so that the momentum generated by the conference can continue on into the future. At the very least, it is clear at this point that the intuition shared by the organizers when we originally applied to hold the conference, that there was something connecting these disparate parts of mathematics, is being born out by the mathematics currently being produced as a direct result of the AIM meeting.