The 3D Euler and 2D surface quasi-geostrophic equations

organized by
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Workshop Summary

This workshop focused on the fundamental problem of whether classical solutions of the 3D Euler and surface quasi-geostrophic (SQG) equations can develop finite time singularities and related issues. During the week-long (April 6-10, 2009) workshop, these issues were extensively studied and investigated through several quite different approaches including modern PDE methods, methods from harmonic analysis and numerical computations. This workshop aimed at developing new strategies and effective tools by combining the advantages of these different approaches.

This report summarizes the daily activities of the workshop. These activities were so arranged that lectures and discussions on similar approaches were accommodated within a close time range.

The first day was devoted to the Caffarelli-Vasseur approach (De Giorgi type argument) on the SQG equation with critical or supercritical dissipation,

$$\partial_t \theta + u \cdot \nabla \theta + \kappa(-\Delta)^\alpha \theta = 0, \tag{1}$$

$$u = \mathcal{R}^\perp \theta \equiv (-\mathcal{R}_2, \mathcal{R}_1)\theta, \tag{2}$$

where $\kappa \geq 0$, $0 < \alpha \leq \frac{1}{2}$ and $\mathcal{R}_i$ denotes the 2D Riesz transforms. Wu gave an overview of the most recent results on the global regularity issue concerning the SQG equation. Vasseur's lecture presented the main results in the work of Caffarelli and Vasseur [CV], in which the global regularity of Leray-Hopf weak solutions to the SQG equation with critical dissipation is established. In particular, Vasseur described in some detail the major steps of the De Giorgi type argument.

The issue that remains open is how to completely solve the global regularity problem for the supercritical case. Part of the Caffarelli-Vasseur approach can be suitably modified to handle the supercritical case. As shown in the work of Constantin and Wu [CW2], any Leray-Hopf weak solution in the supercritical case is still L^∞ for any $t > 0$, and any L^∞ solution with the corresponding velocity field $u \in C^{1-2\alpha}$ is actually Hölder continuous. In another paper [CW1], Constantin and Wu showed that any solution of the SQG equation with Hölder continuity C^δ for $\delta > 1 - 2\alpha$ is actually smooth. Dong and Pavlović [DP] extended the work of Constantin and Wu to cover the case $\delta = 1 - 2\alpha$. It remains open whether or not a weak solution with regularity between L^∞ and C^δ for $\delta < 1 - 2\alpha$ can blow up in a finite time.

During the afternoon discussion session, Vasseur suggested that we employ some non-linear functional spaces involving both spatial and temporal variables in order to further the research on the supercritical case. In one of his recent papers on the estimates of higher derivatives for the 3D Navier-Stokes equations, Vasseur obtained estimates for $\nabla^k u$ in the space-time space $L^p(\mathbf{R}^3 \times (0,T))$, where u represents a solution of the 3D Navier-Stokes equations and k and p satisfy $4/p > k+1$ [Va].

Silvestre also lectured on his very recent work on the eventual regularity of the slightly supercritical SQG equation [Si]. This paper followed the framework of Caffarelli and Vasseur [CV], but it takes into account the exact structure of the nonlinear term. The precise statement of his major result is that given any $T > 0$ there is a $\delta > 0$ such that if $\alpha = \frac{1}{2}(1-\epsilon)$ with $\epsilon \leq \delta$, then the solution is Hölder continuous at time T and becomes smooth thereafter. During the discussion session, the possibility of extending his work to the supercritical case with a fixed $\alpha < \frac{1}{2}$ was discussed.

The second day focused on various approaches involving harmonic analysis. Kiselev first explained the method of modulus of continuity in his work with Nazarov and Volberg [KNV], which successfully resolved the global regularity issue of the critical SQG equation in periodic domains. In addition, Kiselev also talked about a new result indicating the instability of solutions to the SQG equation without dissipation. More precisely, it states that, given any $A > 0$, there exists an initial data θ_0 with $\|\theta_0\|_{H^{11}} < 1$ such that the corresponding solution θ of the inviscid SQG equation (namely (1-2) with $\kappa = 0$) satisfies

$$\limsup_{t\to\infty} \|\theta(\cdot,t)\|_{H^{11}} > A.$$

A. Cordoba's lecture reviewed the progress on the harmonic analysis tools developed by the group of Córdoba's and the applications of these tools to several classes of one-dimensional models of the SQG equation. One such class is

$$\partial_t\theta + \delta\partial_x(H(\theta)\theta) + (1-\delta)H(\theta)\partial_x\theta = 0,$$

where H denotes the Hilbert transform and $\delta \in [0,1]$. This class of 1D models possesses finite-time blowup solutions ([CCCF]). Small non-local dissipation can not prevent finite-time singularities as evident in the equation

$$\partial_t\theta + (H\theta)\theta_x = -\kappa(-\Delta)^{\frac{\gamma}{2}}\theta$$

with $0 \leq \gamma < 1$ ([LR]), although $\gamma > 1$ eliminates finite-time singular solutions ([CCF]).

In the afternoon discussion session, shell models of the 3D Euler and Navier-Stokes equations were examined. In particular, Friedlander described the results in the papers ([CFP],[FP]). Shell models in these papers share many important properties with the 3D Euler and Navier-Stokes equations such as the property conjectured by Onsager [CFP]. Many participants involved in active discussions on how much light these models can shed on the global regularity issue concerning the 3D Euler and Navier-Stokes equations. Also during this discussion session, Constantin shared with the participants an interesting inequality for the Boussinesq type equations

$$u_t + u\cdot\nabla u + \nabla p + \nu L\, u = g\,\theta, \tag{3}$$
$$\theta_t + u\cdot\nabla\theta = 0, \tag{4}$$

where L denotes a positive operator such as $-\Delta$. Then u satisfies

$$\frac{d}{dt}\left[\int |X(a,t)-g\,\theta_0|^2\,da+\int |(u(X(a,t))|^2\,da\right]=-\nu\int u\,L\,u(X(a,t))\,da\le 0,$$

where X denotes the particle trajectory determined by u and θ_0 the initial data for θ. This inequality especially implies the decay of $\int |X(a,t)-g\,\theta_0|^2\,da$.

The third day of this workshop covers recent computational results on the 3D Euler and SQG equations. Ohkitani reported his most recent numerical experiments on the SQG equation corresponding to the initial data

$$\theta(x,0)=\sin x_1\,\sin x_2+\cos x_2 \tag{5}$$

and

$$\theta(x,0)=\cos(2x_1)\cos x_2+\sin x_1\sin x_2+\cos(2x_1)\sin(3x_2). \tag{6}$$

For the dissipative SQG equation, he computed solutions of (1-2) for $\alpha=0.4$, 0.5 and 0.6. His results appear to indicate that the behavior of solutions for these different α's are similar. The criticality of $\alpha=0.5$ does not show up in these computations. In addition, after some time, the solutions appear to settle down to oscillatory steady-state solutions.

Now the issue is whether or not $\alpha=0.5$ is really critical. During the discussion sessions, Hou and Ohkitani tend to believe that $\alpha=0.5$ is an artifact, but more computations are needed in order to make a conclusion.

Hou presented his recent results for the 3D Euler and SQG equations. In particular, he detailed several computational results aiming at the global regularity issue. Numerical solutions of the 3D Euler equations starting with the same initial data used previously by R. Kerr [Ke] do not appear to indicate any finite-time singularity [HouLR]. The theoretical geometric criteria of [DHY] seem to fortify this point. The SQG equation is parallel to the 3D Euler equations and similar conclusions can be drawn for the SQG equation [DHLY]. To completely resolve the global regularity issue via the geometric approach, Hou suggested that we study the evolution of the curvature of the vortex lines (level curves in the case of SQG equation) in the region where the vorticity is comparable to the maximal vorticity. In addition, Hou also talked about two model equations of the 3D axi-symmetric Navier-Stokes equations that have been derived for the purpose of understanding the role of the convection term in the global regularity issue ([HouLC],[HouLZ]).

In the afternoon session, Brachet informed the participants of his computational results on the spectrally-truncated 3D Euler equations. These numerical results show that the truncated Euler equations with large truncation wavenumber has a long-lasting transients that behave like those of the Navier-Stokes equations with large Reynolds number.

Also in this afternoon session, Titi explained in plain words the fundamental mechanism behind the successful proofs of several well-known theorems. First he explained how A. Babin, A. Mahalov and B. Nicolaenko [BMN] were able to establish the global regularity of the 3D rotating Navier-Stokes equations by emphasizing the effects of rotation as in the 1D complex-valued Burgers type equations,

$$u_t+u\,u_z=i\,\Omega\,u$$

which admits global solutions for large real parameter Ω. He also explained the resonance issue that naturally arises in the 3D rotating Navier-Stokes equations by using the real KdV equation as an example. He made the point that fast oscillations actually regularize solutions and thus make it possible to obtain global results in certain special cases of the 3D Navier-Stokes equations. This point is useful in understanding the global results for the 3D incompressible anisotropic Navier-Stokes equations such as the case when there is dissipation only in the horizontal direction but fast oscillation in the vertical direction. Finally, Titi explained the key points of the proof in the work with Cao for the global well-posedness of the 3D viscous primitive equations [CT].

The fourth day was intended to explore the global regularity issue by considering the non-uniqueness of weak solutions. De Lellis presented his recent work on the Euler equations as a differential inclusion ([DeS1],[DeS2]). The major results states that there are bounded and compactly supported divergence-free vector field such that the n-D Euler equations ($n \geq 2$) have infinitely many weak solutions that satisfy either strong or weak local energy inequality or both. All participants show great interest in such weak solutions and their constructions. One particular issue that was discussed is whether or not so constructed weak solutions have a physical meaning.

Gancedo gave the second talk, which focused on the contour dynamics of the 2D incompressible flows with different densities and viscosities. He presented local-existence in Sobolev spaces when, initially, the difference of the gradients of the pressure in the normal direction has the proper sign, an assumption which is also known as the Rayleigh-Taylor condition. In the case where the viscosities are the same, the character of the interphase as the graph of a function is preserved and in this fact has been used in [CG] to prove local existence and a maximum principle, in the stable case, together with ill-possednes in the unstable situation.

In the afternoon discussion session, Lopes explained several recent results on the vanishing viscosity limits of the 2D Navier-Stokes equations in bounded domains with Navier boundary conditions. Mazzucato explained the work of Vishik on the global regularity of solutions in the Besov space $B^{1+\frac{2}{p}}_{p,1}(1 < p < \infty)$ and the major idea of the Besov space technics applied to the 3D Euler equations in Lagrangian coordinates [Vi].

The last day of the workshop started with a presentation given by Chae. He detailed his work on the non-existence of asymptotically self-similar solutions and on several new estimates on solutions of the Euler and Navier-Stokes equations obtained through the generalized self-similarity transforms [Cha]. Constantin, Hou, Titi and several other participants discussed the potential implications of these estimates on the global regularity issue. For the rest of the day, participants automatically divided into small groups discussing and working on some of the open issues covered in the lectures and discussion sessions.

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