1. The Riemann zeta function

The Riemann zeta-function is defined by the series

$$\zeta(s) = 1 + \frac{1}{2^s} + \frac{1}{3^s} + \dots. \tag{1.1}$$

The series converges in the half-plane where the real part of s is larger than 1. The zeta-function can also be expressed as an infinite product over prime numbers

$$\zeta(s) = \frac{2^s}{2^s - 1} \cdot \frac{3^s}{3^s - 1} \cdot \frac{5^s}{5^s - 1} \dots. \tag{1.2}$$

This formula gives the hint that the zeta-function is connected with prime numbers. Bernhard Riemann proved in 1859 that the zeta-function also satisfies a functional equation:

$$\pi^{-s/2}\Gamma(s/2)\zeta(s) = \pi^{-(1-s)/2}\Gamma((1-s)/2)\zeta(1-s) \tag{1.3}$$

where $\pi = 3.14159\dots$ and $\Gamma(s) = (s-1)!$ is Euler's Gamma-function. Riemann also made the famous conjecture that is now known as Riemann's Hypothesis.

2. L-functions

L-functions are functions which enjoy properties similar to the Riemann zeta-function. They can be expressed as series, called Dirichlet series

$$L(s) = 1 + \frac{a_2}{2^s} + \frac{a_3}{3^s} + \frac{a_4}{4^s} + \dots. \tag{2.1}$$

They have Euler products

$$L(s) = \left(1 + \frac{a_2}{2^s} + \frac{a_4}{4^s} + \frac{a_8}{8^s} + \dots\right)\left(1 + \frac{a_3}{3^s} + \frac{a_9}{9^s} + \dots\right)\left(1 + \frac{a_5}{5^s} + \frac{a_{25}}{25^s} + \dots\right)\dots \tag{2.2}$$

They also have functional equations

$$F(s) = \left(\frac{q}{\pi^d}\right)^{s/2}\Gamma(s/2 + r_1)\Gamma(s/2 + r_2)\dots\Gamma(s/2 + r_d)L(s) = F(1-s). \tag{2.3}$$

Here q is an integer, called the *level*, d is called the degree, and the numbers r_d are called the Langland's parameters. For the Riemann zeta-function, the level is 1, the degree is 1, and the Langland's parameter is 0. If the Langland's parameters are rational or algebraic numbers, then we call the L-function "algebraic." If these numbers are transcendental, then we call the L-function transcendental.

3. L-functions and Fermat's Last Theorem

There are lots of examples of a square being separated as a sum of two other squares, such as $5^2 = 3^2 + 4^2$ or $13^2 = 5^2 + 12^2$. Triples of these, such as $(3, 4, 5)$ or $(5, 12, 13)$ are called Pythagorean triples and form the sides of a right triangle. Fermat asserted in 1626 that by contrast a cube could never be the sum of two cubes, a fourth power could not be the sum of two fourth powers, and so on. In other words, one can never have $z^n = x^n + y^n$ where x, y, and z are positive whole numbers, for $N \geq 3$. This claim came to be known as Fermat's Last Theorem, even though it wasn't a theorem until, more than 350 years later, Andrew Wiles proved that Fermat was correct. The crux of his proof was showing that certain Dirichlet series associated with elliptic curves were in fact algebraic L-functions of degree 2.

4. The Riemann Hypothesis

The Riemann Hypothesis is the assertion that all of the non-real zeros of $\zeta(s)$ are on the vertical line which passes through the number $1/2$ in the complex plane. Another formulation which shows the connection with primes is that the number of primes less than a large number N is given by

$$\frac{1}{\ln 2} + \frac{1}{\ln 3} + \frac{1}{\ln 4} + \cdots + \frac{1}{\ln N} + E(N) \qquad (4.1)$$

where $E(N)$ is no larger than $\sqrt{N}\log N$. Thus, the Riemann Hypothesis is the assertion that the primes are very regularly distributed when seen from a large scale.