DEFINITIONS

$$U = (u_{jk}) \qquad U^t = (u_{kj}) \qquad U^* = (\overline{u_{kj}})$$

$$\begin{aligned} U(N) &= \{U \text{ an } N \times N \text{ matrix } : UU^* = I\} \\ SO(N) &= \{R \in U(N) : RR^t = I, \det R = 1\} \\ USp(2N) &= \{P \in U(2N) : PZP^t = Z\} \end{aligned}$$

$$Z = \begin{pmatrix} 0 & I_N \\ -I_N & 0 \end{pmatrix}$$

EIGENVALUES

$$
\begin{array}{rcc}
U(N): & e^{i\theta_1}, \ldots, e^{i\theta_N} & 0 \le \theta_j < 2\pi \\
SO(2N): & e^{\pm i\theta_1}, \ldots, e^{\pm i\theta_N} & 0 \le \theta_j \le \pi \\
SO(2N+1): & 1, e^{\pm i\theta_1}, \ldots, e^{\pm i\theta_N} & 0 \le \theta_j \le \pi \\
USp(2N): & e^{\pm i\theta_1}, \ldots, e^{\pm i\theta_N} & 0 \le \theta_j \le \pi
\end{array}
$$

We will use dU_N as a shorthand for the Haar measure on $U(N)$;
dP_N for the Haar measure on $USp(2N)$;
dR_N^+ for the Haar measure on $SO(2N)$ and dR_N^- for $SO(2N+1)$

So R is for "R-thogonal"

HAAR MEASURES

$$dU_N = \frac{1}{(2\pi)^N N!} \prod_{1\leq j<k\leq N} |e^{i\theta_k} - e^{i\theta_j}|^2 d\theta_1 \ldots d\theta_N$$

$$dR_N^+ = \frac{2^{(N-1)^2}}{\pi^N N!} \prod_{1\leq j<k\leq N} (\cos\theta_k - \cos\theta_j)^2 d\theta_1 \ldots d\theta_N$$

$$dR_N^- = \frac{2^{N^2}}{\pi^N N!} \prod_{1\leq j<k\leq N} (\cos\theta_k - \cos\theta_j)^2 \prod_{n=1}^{N} \sin^2 \frac{\theta_n}{2} d\theta_n$$

$$dP_N = \frac{2^{N^2}}{\pi^N N!} \prod_{1\leq j<k\leq N} (\cos\theta_k - \cos\theta_j)^2 \prod_{n=1}^{N} \sin^2 \theta_n d\theta_n$$

WEYL INTEGRATION FORMULAE

If $f(R)$ is a class function on $SO(2N)$ (i.e. it depends only on the set of eigenvalues of R), then

$$\int_{SO(2N)} f(R)\, dR_N^+$$

$$= \frac{2^{(N-1)^2}}{\pi^N N!} \int_{[0,\pi]^N} f(\theta_1, \ldots, \theta_N) \prod_{1 \le j < k \le N} (\cos\theta_k - \cos\theta_j)^2 d\theta_1 \ldots d\theta_N .$$

VANDERMONDE DETERMINANTS

For complex numbers $(x_1, \ldots, x_N)$ let

$$\Delta(x_1, \ldots, x_N) = \det_{N \times N} \left(x_k^{j-1}\right)$$

Then

$$\Delta(x_1, \ldots, x_N) = \prod_{1 \le j < k \le N} (x_k - x_j).$$

Proof: Both sides are homogeneous polynomials of total degree $N(N-1)/2$ which vanish whenever $x_j = x_k$. This fact identifies the two sides up to a constant factor. The coefficient of $x_N^{N-1} x_{N-1}^{N-2} \ldots x_2$ is 1 in both expressions.

VANDERMONDES CONT'D

Observe that

$$\prod_{1\le j<k\le N} |e^{i\theta_k} - e^{i\theta_j}|^2 = |\Delta(e^{i\theta_1},\ldots,e^{i\theta_N})|^2 = \left(\det_{N\times N}(e^{i(j-1)\theta_k})\right)^2$$

and

$$\prod_{1\le j<k\le N} (\cos\theta_k - \cos\theta_j)^2 = \Delta(\cos\theta_1,\ldots,\cos\theta_N)^2 = \left(\det_{N\times N}(\cos^{j-1}\theta_k)\right)^2$$

ANDREIF'S IDENTITY

For any interval J and integrable functions ϕ_j and ψ_j:

$$\frac{1}{N!}\int_{J^N} \det_{N\times N}(\phi_j(\theta_k)) \det_{N\times N}(\psi_j(\theta_k))\ d\theta_1 \ldots d\theta_N$$

$$=$$

$$\det_{N\times N}\Big(\int_J \phi_j(\theta)\psi_k(\theta)\ d\theta\Big)$$

MASS OF THE HAAR MEASURE OF U(N)

Using

$$\phi_j(\theta) = e^{i(j-1)\theta}$$

we see that

$$\begin{aligned}
\int_{[0,2\pi]^N} dU_N &= \int_{[0,2\pi]^N} |\det_{N\times N}(e^{i(j-1)\theta_k})|^2 \frac{d\theta_1 \ldots\ d\theta_N}{N!(2\pi)^N} \\
&= \frac{1}{(2\pi)^N} \det_{N\times N} \Big(\int_0^{2\pi} e^{i(j-1)\theta} e^{-i(k-1)\theta}\ d\theta \Big) \\
&= \frac{1}{(2\pi)^N} \det_{N\times N} \big(2\pi I\big) = 1
\end{aligned}$$

This verifies that the total mass of the Haar measure of $U(N)$ is 1.

ORTHOGONAL POLYNOMIALS

$$\begin{aligned} T_n(\cos\theta) &:= \cos n\theta \\ U_n(\cos\theta) &:= \frac{\sin(n+1)\theta}{\sin\theta} \\ V_n(\cos\theta) &:= \frac{\sin(n+\frac{1}{2})\theta}{\sin\frac{\theta}{2}} \end{aligned}$$

$$T_n(x) = 2^{n-1}x^n + \dots \quad U_n(x) = 2^n x^n + \dots \quad V_n(x) = 2^n x^n + \dots$$

$$T_0^*(x) := 1 \qquad T_n^*(x) := \sqrt{2}T_n(x) \quad (n > 1)$$

T^*, U, V orthogonal on $[0,\pi]$ w.r.t. $d\theta$, $\sin^2\theta\ d\theta$, $\sin^2\frac{\theta}{2}\ d\theta$

Each vector has the same norm: $||T_j^*||^2 = \pi, ||U_j||^2 = \frac{\pi}{2}, ||V_j||^2 = \frac{\pi}{2}$.

REWRITING THE VANDERMONDE

By elementary row operations

$$\begin{aligned}
&\Delta(\cos\theta_1, \dots, \cos\theta_N) \\
&\quad = 2^{-(N-1)^2/2} \det_{N\times N}\left(T^*_{j-1}(\cos\theta_k)\right) \\
&\quad = 2^{-N(N-1)/2} \det_{N\times N}\left(U_{j-1}(\cos\theta_k)\right) \\
&\quad = 2^{-N(N-1)/2} \det_{N\times N}\left(V_{j-1}(\cos\theta_k)\right)
\end{aligned}$$

HAAR MEASURES AS SQUARES OF DETS

$$
\begin{aligned}
dU_N &= \frac{1}{(2\pi)^N N!}\left| \det_{N\times N}\left(e^{i(j-1)\theta_k}\right)\right|^2 d\theta_1 \ldots d\theta_N \\
dR_N^+ &= \frac{1}{\pi^N N!}\left(\det_{N\times N}\left(T_{j-1}^*(\cos\theta_k)\right)\right)^2 d\theta_1 \ldots d\theta_N \\
dR_N^- &= \frac{2^N}{\pi^N N!}\left(\det_{N\times N}\left(V_{j-1}(\cos\theta_k)\right)\right)^2 \prod_{n=1}^{N} \sin^2\frac{\theta_n}{2}\ d\theta_n \\
dP_N &= \frac{2^N}{\pi^N N!}\left(\det_{N\times N}\left(U_{j-1}(\cos\theta_k)\right)\right)^2 \prod_{n=1}^{N} \sin^2\theta_n\ d\theta_n
\end{aligned}
$$

MASS OF dR_N^+

$$\int_{[0,\pi]^N} dR_N^+ = \frac{1}{\pi^N N!} \det_{N\times N}\left(\int_0^\pi T_{j-1}^*(\cos\theta)T_{k-1}^*(\cos\theta)\ d\theta\right) = 1,$$

since

$$\int_0^\pi T_{j-1}^*(\cos\theta)T_{k-1}^*(\cos\theta)\ d\theta = \pi\delta_{j,k}$$

GENERALIZED ANDREIF

$$\frac{1}{N!}\int_{J^N}\prod_{i=1}^{N} f(\theta_i)\,\det_{N\times N}(\phi_j(\theta_k))\,\det_{N\times N}(\psi_j(\theta_k))\;d\theta_1\ldots d\theta_N$$

$$= \det_{N\times N}\Big(\int_J f(\theta)\phi_j(\theta)\psi_k(\theta)\;d\theta\Big)$$

MASS OF dR_N^- AND dP_N

$$\begin{aligned}\int_{SO(2N+1)} dR_N^- &= \frac{2^N}{\pi^N} \det_{N\times N}\left(\int_0^\pi \; V_{j-1}(\cos\theta)\; V_{k-1}(\cos\theta)\sin^2\frac{\theta}{2}\; d\theta\right) \\ &= 1\end{aligned}$$

and

$$\begin{aligned}\int_{USp(2N)} dP_N &= \frac{2^N}{\pi^N} \det_{N\times N}\left(\int_0^\pi \; U_{j-1}(\cos\theta)\; U_{k-1}(\cos\theta)\sin^2\theta\; d\theta\right) \\ &= 1\end{aligned}$$

since the integrals are $\frac{\pi}{2}$ when $j = k$ and 0 otherwise.

PROOF OF ANDREIF'S IDENTITY

$$\int_{J^N} \det_{N\times N}\big(\phi_j(\theta_k)\big) \det_{N\times N}\big(\psi_j(\theta_k)\big) d\theta_1 \ldots\ d\theta_N$$

$$= \int_{J^N} \sum_{\sigma} \operatorname{sgn}(\sigma) \prod_{j=1}^{N} \phi_j(\theta_{\sigma j}) \sum_{\tau} \operatorname{sgn}(\tau) \prod_{k=1}^{N} \psi_k(\theta_{\tau k}) \prod_{i=1}^{N} d\theta_i$$

$$\underset{\tau\to\sigma\tau}{=} \int_{J^N} \sum_{\sigma,\tau} \operatorname{sgn}(\tau) \prod_{j,k=1}^{N} \phi_j(\theta_{\sigma j}) \psi_k(\theta_{\sigma\tau k}) \prod_{i=1}^{N} d\theta_i$$

$$\underset{k\to\tau^{-1}k}{=} \int_{J^N} \sum_{\sigma,\tau} \operatorname{sgn}(\tau) \prod_{j,k=1}^{N} \phi_j(\theta_{\sigma j}) \psi_{\tau^{-1}k}(\theta_{\sigma k}) \prod_{i=1}^{N} d\theta_i$$

$$
\begin{aligned}
&= \int_{J^N} \sum_{\sigma,\tau} \operatorname{sgn}(\tau) \prod_{j=1}^{N} \phi_j(\theta_{\sigma j}) \psi_{\tau^{-1} j}(\theta_{\sigma j}) \ \prod_{i=1}^{N} d\theta_i \\
&\underset{\tau \to \tau^{-1}}{=} \int_{J^N} \sum_{\sigma,\tau} \operatorname{sgn}(\tau) \prod_{j=1}^{N} \phi_j(\theta_{\sigma j}) \psi_{\tau j}(\theta_{\sigma j}) \ \prod_{i=1}^{N} d\theta_i \\
&= \sum_{\sigma,\tau} \operatorname{sgn}(\tau) \prod_{j=1}^{N} \int_J \phi_j(\theta) \psi_{\tau j}(\theta) \ d\theta \\
&= N! \sum_{\tau} \operatorname{sgn}(\tau) \prod_{j=1}^{N} \int_J \phi_j(\theta) \psi_{\tau j}(\theta) \ d\theta \\
&= N! \det_{N \times N} \Big(\int_J \phi_j(\theta) \psi_k(\theta) \ d\theta \Big).
\end{aligned}
$$

TRANSPOSING LEMMA

$$\det_{N\times N}\left(\phi_{j-1}(x_k)\right)\det_{N\times N}\left(\psi_{j-1}(x_k)\right)=\det_{N\times N}\Big(\sum_{n=1}^{N}\phi_{n-1}(x_j)\psi_{n-1}(x_k)\Big).$$

This identity just follows by using the fact that the determinant of a matrix and its transpose are the same, and matrix multiplication:

$$\begin{aligned}&\det_{N\times N}\left(\phi_{j-1}(x_k)\right)\det_{N\times N}\left(\psi_{j-1}(x_k)\right)\\&=\det_{N\times N}\left(\phi_{j-1}(x_n)\right)_{j,n}\det_{N\times N}\left(\psi_{n-1}(x_k)\right)_{n,k}\\&=\det\Big(\sum_{n=1}^{N}\phi_{n-1}(x_j)\psi_{n-1}(x_k)\Big)_{j,k}\end{aligned}$$

TRANSPOSING THE SQUARE DETS IN THE MEASURES

$$\left| \det_{N\times N} e^{i(j-1)\theta_k} \right|^2 = \det_{N\times N} \left(\sum_{n=1}^{N} e^{i(n-1)\theta_j} e^{-i(n-1)\theta_k} \right)$$

$$\left(\det_{N\times N} \left(T^*_{j-1}(\cos\theta_k)\right) \right)^2 = \det_{N\times N} \left(\sum_{n=1}^{N} T^*_{n-1}(\cos\theta_j) T^*_{n-1}(\cos\theta_k) \right)$$

$$= \det_{N\times N} \left(1 + 2 \sum_{n=1}^{N-1} \cos n\theta_j \cos n\theta_k \right)$$

$$\left(\det_{N\times N} \left(U_{j-1}(\cos\theta_k)\right) \right)^2 \prod_{n=1}^{N} \sin^2\theta_n = \det_{N\times N} \left(\sum_{n=1}^{N} \sin n\theta_j \sin n\theta_k \right)$$

$$\left(\det_{N\times N} \left(V_{j-1}(\cos\theta_k)\right) \right)^2 \prod_{n=1}^{N} \sin^2\frac{\theta_n}{2} = \det_{N\times N} \left(\sum_{n=1}^{N} \sin(n-\tfrac{1}{2})\theta_j \sin(n-\tfrac{1}{2})\theta_k \right)$$

HAAR MEASURES AS DETS

$$
\begin{aligned}
dU_N &= \frac{1}{(2\pi)^N N!} \det_{N\times N} \left(\sum_{n=1}^{N} e^{i(n-1)\theta_j} e^{-i(n-1)\theta_k} \right) d\theta_1 \dots d\theta_N \\
dR_N^{+} &= \frac{1}{\pi^N N!} \det_{N\times N} \big(1 + 2 \sum_{n=1}^{N-1} \cos n\theta_j \cos n\theta_k \big) d\theta_1 \dots d\theta_N \\
dR_N^{-} &= \frac{2^N}{\pi^N N!} \det_{N\times N} \big(\sum_{n=1}^{N} \sin(n - \tfrac{1}{2})\theta_j \sin(n - \tfrac{1}{2})\theta_k \big) d\theta_1 \dots d\theta_N \\
dP_N &= \frac{2^N}{\pi^N N!} \det_{N\times N} \big(\sum_{n=1}^{N} \sin n\theta_j \sin n\theta_k \big) d\theta_1 \dots d\theta_N
\end{aligned}
$$

TRIGONOMETRIC SUMS

$$S_N(\theta) := \frac{\sin \frac{N\theta}{2}}{\sin \frac{\theta}{2}}$$

$$\sum_{n=1}^{N} e^{i(n-1)\theta_j} e^{-i(n-1)\theta_k} = e^{-iN(\theta_k-\theta_j)/2} S_N(\theta_k - \theta_j)$$

$$1 + 2\sum_{n=1}^{N-1} \cos nx \cos ny = (S_{2N-1}(y-x) + S_{2N-1}(y+x))/2$$

$$2\sum_{n=1}^{N} \sin nx \sin ny = (S_{2N+1}(y-x) - S_{2N+1}(y+x))/2$$

$$2\sum_{n=1}^{N} \sin(n-\tfrac{1}{2})x \sin(n-\tfrac{1}{2})y = (S_{2N}(y-x) - S_{2N}(y+x))/2$$

ALTERNATE FORMULAS FOR HAAR MEASURES

$$
\begin{aligned}
S_{U,N}(x,y): &= S_N(y-x) \\
S_{R^+,N}(x,y): &= \frac{1}{2}\left(S_{2N-1}(y-x)+S_{2N-1}(y+x)\right) \\
S_{R^-,N}(x,y): &= \frac{1}{2}\left(S_{2N}(y-x)-S_{2N}(y+x)\right) \\
S_{P,N}(x,y): &= \frac{1}{2}\left(S_{2N+1}(y-x)-S_{2N+1}(y+x)\right)
\end{aligned}
$$

ALTERNATE FORMULAS CONT'D

$$
\begin{aligned}
dU_N &= \det_{N\times N}(S_{U,N}(\theta_j,\theta_k))\frac{d\theta_1\ldots d\theta_N}{(2\pi)^N N!} \\
dR_N^+ &= \det_{N\times N}(S_{R+,N}(\theta_j,\theta_k))\frac{d\theta_1\ldots d\theta_N}{\pi^N N!} \\
dR_N^- &= \det_{N\times N}(S_{R-,N}(\theta_j,\theta_k))\frac{d\theta_1\ldots d\theta_N}{\pi^N N!} \\
dP_N &= \det_{N\times N}(S_{P,N}(\theta_j,\theta_k))\frac{d\theta_1\ldots d\theta_N}{\pi^N N!}
\end{aligned}
$$

GAUDIN'S LEMMA

Suppose that we have a function f and a measurable set J such that

$$\int_J f(x,\theta)f(\theta,y)\;d\theta = Cf(x,y)$$

for all x and y where $C = C(J,f)$ is a constant, and

$$\int_J f(x,x)\;dx = D.$$

Then,

$$\int_J \det_{M\times M}\big(f(\theta_j,\theta_k)\big)\;d\theta_M = (D-(M-1)C)\det_{M-1}\big(f(\theta_j,\theta_k)\big)$$

GAUDIN'S LEMMA AND $S_{G,N}$

Let G denote R^+, R^-, or P. Then

$$\int_{[0,\pi]} S_{G,N}(x,\theta)S_{G,N}(\theta,y)\ d\theta = \pi S_{G,N}(x,y)$$

$$\int_0^{\pi} S_{G,N}(x,x)\ dx = \pi N$$

Also

$$\int_{[0,2\pi]} S_{U,N}(x,\theta)S_{U,N}(\theta,y)\ d\theta = 2\pi S_{U,N}(x,y)$$

$$\int_0^{2\pi} S_{U,N}(x,x)\ dx = 2\pi N$$

GAUDIN'S LEMMA AND $S_{G,N}$

For $G = R^+, R^-, P$:

$$\int_{[0,\pi]} \det_{M\times M} S_{G,N}(\theta_j,\theta_k)\, d\theta_M = \pi(N-(M-1)) \det_{M-1} S_{G,N}(\theta_j,\theta_k)$$

Applying this with $M = N$, then $M = N-1$, ..., then $M = n+1$:

$$\int_{[0,\pi]^{N-n}} \det_{N\times N} S_{G,N}(\theta_j,\theta_k)\, d\theta_{n+1}\ldots d\theta_N = \pi^{N-n}(N-n)! \det_{n\times n} S_{G,N}(\theta_j,\theta_k)$$

Similarly,

$$\int_{[0,2\pi]^{N-n}} \det_{N\times N} S_{U,N}(\theta_j,\theta_k)\, d\theta_{n+1}\ldots d\theta_N = (2\pi)^{N-n}(N-n)! \det_{n\times n} S_{U,N}(\theta_j,\theta_k)$$

PROOF OF REPRODUCING PROPERTY FOR $S_{P,N}$

$$S_{P,N}(x,y) = 2\sum_{n=1}^{N} U_{n-1}(\cos x)U_{n-1}(\cos y)\sin x \sin y$$

$$\int_0^{\pi} S_{P,N}(x,\theta)S_{P,N}(\theta,y)\ d\theta = 4\sin x \sin y$$

$$\times \int_0^{\pi} \sum_{m,n=0}^{N-1} U_n(\cos x)U_n(\cos\theta)U_m(\cos\theta)U_m(\cos y)\sin^2\theta\ d\theta$$

$$= 4\sin x \sin y \frac{\pi}{2} \sum_{n=0}^{N-1} U_n(\cos x)U_n(\cos y)$$

$$= 2\pi \sum_{n=1}^{N} U_{n-1}(\cos x)U_{n-1}(\cos y)\sin x \sin y = \pi S_{P,N}(x,y)$$

USING GAUDIN'S LEMMA TO COMPUTE n-LEVEL DENSITY

Let $B \subset \{1, \ldots, N\}$, $|B| = n$. Let f be a symmetric function of n vbles. Let $f(\theta_B) = f(\theta_{b_1}, \ldots, \theta_{b_n})$ where $B = \{b_1, \ldots, b_n\}$. Then

$$\int_{U(N)} \sum_{\substack{B \subset [1,N] \\ |B|=n}} f(\theta_B) dU_N$$

$$= \frac{1}{(2\pi)^n n!} \int_{[0,2\pi]^n} f(\theta_1, \ldots, \theta_n) \det_{n\times n}(S_{U,N}(\theta_j, \theta_k)) d\theta_1 \ldots d\theta_n$$

We call $\det_{n\times n}(S_{U,N}(\theta_j, \theta_k))$ the n-level density function for $U(N)$.

THE OTHER n-LEVEL DENSITIES

Let $G = R^+, R^-$, or P. Then

$$\int_{G,N} \sum_{\substack{B\subset[1,N]\\|B|=n}} f(\theta_B)dG_N$$

$$= \frac{1}{\pi^n n!}\int_{[0,\pi]^n} f(\theta_1,\ldots,\theta_n) \det_{n\times n}(S_{G,N}(\theta_j,\theta_k))d\theta_1\ldots d\theta_n$$

We call $\det_{n\times n}(S_{G,N}(\theta_j,\theta_k))$ the n-level density function for G, N.

LARGE N LIMIT OF THE n-LEVEL DENSITY

Suppose that we have an infinite sequence $X : 0 \leq x_1, x_2, \ldots$ whose average spacing is 1, i.e. $\lim_{N\to\infty}(x_N - x_1)/N = 1$. Suppose that f is a symmetric function of n variables which is rapidly decaying as one leaves the origin. For each subset
$B = \{b_1, \ldots, b_n\} \subset \{1, \ldots, N\}$ of size n we evalate f at the n-tuple of points whose indices are in B. We add all of these up and take the limit as $N \to \infty$. If this limit exists and if, for some K,

$$\lim_{N\to\infty} \sum_{\substack{B\subset[1,N]\\ |B|=n}} f(x_B) = \frac{1}{n!}\int_{R_+^n} f(x_1, \ldots, x_n)K(x_1, \ldots, x_n)\, dx_1 \ldots dx_n$$

for all reasonable f, then we call $K(x_1, \ldots, x_n)$ the n-level density function associated with X.

NORMALIZED EIGENANGLES FOR $U(N)$

For $U \in U(N)$ with eigenvalues $e^{i\theta_1}, \ldots, e^{i\theta_N}$ let $\tilde{U} = \{\frac{N\theta_1}{2\pi}, \ldots, \frac{N\theta_N}{2\pi}\}$. These N numbers are all contained in $[0, N]$. Letting $\tilde{\theta} = N\theta/(2\pi)$ we calculate

$$\lim_{N\to\infty} \int_{U(N)} \sum_{\substack{B\subset[1,N] \\ |B|=n}} f(\tilde{\theta}_B) dU_N$$

$$= \lim_{N\to\infty} \frac{1}{(2\pi)^n n!} \int_{[0,2\pi]^n} f(\tilde{\theta}_1, \ldots, \tilde{\theta}_n) \det_{n\times n} (S_{U,N}(\theta_j, \theta_k)) d\theta_1 \ldots d\theta_n$$

$$= \lim_{N\to\infty} \frac{1}{n!} \int_{[0,N]^n} f(\theta_1, \ldots, \theta_n) \det_{n\times n} \left(\frac{S_N(2\pi(\theta_k - \theta_j))/N}{N} \right) \prod_{j=1}^{n} d\theta_j$$

$$= \frac{1}{n!} \int_{R_+^n} f(\theta_1, \ldots, \theta_n) \det_{n\times n} \big(S(\theta_k - \theta_j)\big) \prod_{j=1}^{n} d\theta_j$$

since $\lim_{N\to\infty} S_N(2\pi x/N)/N = \lim \sin(\pi x)/(N \sin \pi x/N) = S(x)$

NORMALIZED EIGENANGLES FOR $USp(N)$

For $P \in USp(2N)$ with eigenvalues $e^{\pm i\theta_1}, \ldots, e^{\pm i\theta_N}$ and $0 \leq \theta_j \leq \pi$, let $\tilde{P} = \{\frac{N\theta_1}{\pi}, \ldots, \frac{N\theta_N}{\pi}\}$. These N numbers are all contained in $[0, N]$. Letting $\tilde{\theta} = N\theta/\pi$ we calculate

$$\lim_{N\to\infty} \int_{USp(2N)} \sum_{\substack{B\subset[1,N] \\ |B|=n}} f(\tilde{\theta}_B) dP_N$$

$$= \lim_{N\to\infty} \frac{1}{\pi^n n!} \int_{[0,\pi]^n} f(\tilde{\theta}_1, \ldots, \tilde{\theta}_n) \det_{n\times n}(S_{P,N}(\theta_j, \theta_k)) d\theta_1 \ldots d\theta_n$$

$$= \lim_{N\to\infty} \frac{1}{n!} \int_{[0,N]^n} f(\theta_1, \ldots, \theta_n) \det_{n\times n} \left(\frac{S_{P,N}(\pi\theta_j/N, \pi\theta_k/N)}{N} \right) \prod_{j=1}^{n} d\theta_j$$

$$= \frac{1}{n!} \int_{R_+^n} f(\theta_1, \ldots, \theta_n) \det_{n\times n} \big(S(\theta_k - \theta_j) - S(\theta_j + \theta_k) \big) \prod_{j=1}^{n} d\theta_j$$

since $\lim_{N\to\infty} S_{2N+1}(\pi x/N)/(2N) =$ $\lim \sin(\pi x(1 + 1/(2N)))/(2N \sin \pi x/(2N)) = S(x)$

DENSITY FUNCTIONS

$$
\begin{aligned}
n\text{-level density for U}: &= \det_{n\times n} S_U(\theta_j,\theta_k)\text{where} \\
S_U(x,y) &= S(y-x) \\
n\text{-level density for G}: &= \det_{n\times n} S_G(\theta_j,\theta_k)\text{where} \\
S_{R^+}(x,y) &= S(y-x)+S(y+x) \\
S_P(x,y) &= S(y-x)-S(y+x)
\end{aligned}
$$

n-level density for R^-:

$$= \det_{n\times n} S_P(\theta_j,\theta_k) + \sum_{m=1}^{n} \delta(\theta_m) \det_{n-1} S_P^{(m)}(\theta_j,\theta_k)$$

where $S_P^{(m)}$ means the mth row and mth column omitted

1-LEVEL DENSITY FUNCTIONS

$$
\begin{aligned}
U &: \quad 1 \\
R^{+} &: \quad 1 + \frac{\sin 2\pi x}{2\pi x} \\
R^{-} &: \quad \delta(x) + 1 - \frac{\sin 2\pi x}{2\pi x} \\
P &: \quad 1 - \frac{\sin 2\pi x}{2\pi x}
\end{aligned}
$$

LARGE N LIMIT OF THE n-CORRELATION FUNCTION

Suppose that we have an infinite sequence $X : 0 \leq x_1, x_2, \ldots$ whose average spacing is 1, i.e. $\lim_{N\to\infty}(x_N - x_1)/N = 1$. Suppose that f is a symmetric function of n variables which is **translation invariant** and rapidly decaying as one leaves the origin in the direction of a fixed vector not parallel to (1,...,1). For each subset $B = \{b_1, \ldots, b_n\} \subset \{1, \ldots, N\}$ of size n we evalate f at the n-tuple of points whose indices are in B. We add all of these up, **divide by N** and take the limit as $N \to \infty$. If this limit exists and

$$\lim_{N\to\infty} \frac{1}{N} \sum_{\substack{B\subset[1,N]\\ |B|=n}} f(x_B)$$

$$= \frac{1}{(n-1)!} \int_{R_+^{n-1}} f(x_1, \ldots, x_n) L(x_1, \ldots, x_n)\Big|_{x_1=0} \, dx_2 \ldots dx_n$$

for all reasonable f, then we call $L(x_1, \ldots, x_n)$ the n-correlation function associated with X.

n-CORRELATION FOR $U(N)$

$f(\theta_1+\mu,\ldots,\theta_n+\mu) = f(\theta_1,\ldots,\theta_n)$ and $f(0,\theta_2,\ldots,\theta_n)$ is compactly supported on $[0,A]$.

$$Q_N(f) = \int_{U(N)} \sum_{1\le j_1<j_2<\cdots<j_n\le N} f(\tilde{\theta}_{j_1},\ldots,\tilde{\theta}_{j_n})dU_N$$

By Gaudin $Q_N(f) =$

$$= \frac{1}{n!}\int_{[0,N]^n} f(\theta_1,\ldots,\theta_n) \det_{n\times n} \frac{1}{N} S_N(2\pi(\theta_j-\theta_k)/N)d\theta_1\ldots d\theta_n$$

$$= \int_{0\le\theta_1\le\cdots\le\theta_n\le N} f(\theta_1,\ldots,\theta_n) \det_{n\times n} \frac{1}{N} S_N(2\pi(\theta_j-\theta_k)/N)d\theta_1\ldots d\theta_n$$

Let $x_1=\theta_1$, $x_2=\theta_2-\theta_1$, ... $x_n=\theta_n-\theta_1$. Then $Q_N(f) =$

$$\int_0^N \int_{0\le x_2\le\cdots\le x_n\le N-x_1} g(x_1,x_2+x_1,\ldots,x_n+x_1)dx_1\ldots dx_n$$

where $g(x_1,\ldots,x_n) = f(x_1,\ldots,x_n)\det(\frac{S_N(2\pi(x_k-x_j)/N)}{N})$.

n-CORRELATION FOR U, CONT'D

g is translation invariant and $g(0, x_2, \ldots, x_n)$ is compactly supported, so $Q_N(f)$

$$= \int_0^N \int_{0 \le x_2 \le \cdots \le x_n \le N - x_1} g(0, x_2, \ldots, x_n) dx_1 \ldots dx_n$$

$$= \int_{0 \le x_2 \le \cdots \le x_n \le A} g(0, x_2, \ldots, x_n) \int_0^{N-A} dx_1.$$

Thus $\lim_{N \to \infty} \frac{Q_N(f)}{N} =$

$$\frac{1}{(n-1)!} \int_{\mathbf{R}_+^{n-1}} f(x_1, x_2, \ldots, x_n) \det_{n \times n} S(x_j - x_k) \Big|_{x_1 = 0} dx_2 \ldots dx_n.$$

Therefore, $\det_{n \times n} S(x_j - x_k)$ is the n-correlation function for U.

n-CORRELATION FOR P

By Gaudin's Lemma and a change of variables, $Q_N(f)$

$$= \frac{1}{n!} \int_{[0,N]^n} f(\theta_1, \dots, \theta_n)$$

$$\det_{n\times n} \frac{1}{2N} \big(S_{2N-1}(\pi(\theta_k - \theta_j)/N) + S_{2N-1}(\pi(\theta_k + \theta_j)/N)\big) d\theta_1 \dots d\theta_n$$

$$= \int_{0\le\theta_1\le\cdots\le\theta_n\le N} \ldots$$

We make the change of variable $x_1 = \theta_1$, $x_2 = \theta_2 - \theta_1$, ...
$x_n = \theta_n - \theta_1$

$$\int_{0\le x_2\le\cdots\le x_n\le A} f(0, x_2, \dots, x_n)$$

$$\int_0^{N-x_n} \det_{n\times n} \frac{1}{2N} \big(S_{2N-1}(\pi(x_k - x_j)/N)\big|_{x_1=0}$$

$$+S_{2N-1}(\pi(x_k^* + x_j^* + 2x_1)/N)\big) dx_2 \dots dx_n dx_1$$

where $x_j^* = x_j$ if $j \neq 1$ whereas $x_1^* = 0$. Now we claim that

$$\lim_{N\to\infty} \frac{1}{N} \int_0^{N-x_n} \det_{n\times n} \frac{1}{2N} \big(S_{2N-1}(\pi(x_k - x_j)/N)\big|_{x_1=0}$$
$$+S_{2N-1}(\pi(x_k^* + x_j^* + 2x_1)/N)\big) dx_1 = \det_{n\times n} S(x_k - x_j)\big|_{x_1=0}.$$

To see this, note that in the expansion of the determinant there are $n!$ terms each of which is a product of n factors $\frac{1}{2N} S_{2N-1}(\pi(x_k - x_j))|_{x_1=0} + \frac{1}{2N} S_{2N-1}(\pi(x_k^* + x_j^* + 2x_1))$. If we multiply out each term, there are 2^n terms, all but one of which will contain at least one factor with $\frac{1}{2N} S_{2N-1}(\pi(x_k + x_j + 2x_1))$. Any of the terms with at least one factor like this will tend to 0 after integrating with respect to x_1 and dividing by N; for letting

$$c(a,b,N)(x) = \frac{\sin(ax+b)}{N\sin(ax/N + b/N)},$$

it is not difficult to see that $c(a,b,N)(x) \leq \frac{2}{\pi} \frac{\sin(ax+b)}{ax+b}$ provided that $ax + b < \frac{\pi N}{2}$, and $|c(a,b,N)(x)| \leq 1$ for all x and integer N. Therefore, using the fact that $\int_0^B \left(\frac{\sin x}{x}\right)^j \, dx$ is uniformly bounded in B for each fixed j, we see that

$$\frac{1}{N}\int_0^{N-B} \prod_{j=1}^{J} c(a_j, b_j, N)(x) \; dx \to 0$$

as $N \to \infty$ through integers. This leaves only the term with all $\frac{1}{2N} S_{2N-1}(\pi(x_k - x_j))$ factors which tend to $S(x_k - x_j)$ as $N \to \infty$.

n-CORRELATION FUNCTIONS

The n-correlation functions are:

$$
\begin{aligned}
U : &\qquad \det_{n\times n} S(\theta_k - \theta_j) \\
R^+ : &\qquad \det_{n\times n} S(\theta_k - \theta_j) \\
R^- : &\qquad \det_{n\times n} S(\theta_k - \theta_j) \\
P : &\qquad \det_{n\times n} S(\theta_k - \theta_j)
\end{aligned}
$$

NEIGHBOR SPACINGS

If

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} f(x_{n+1} - x_n) = \int_0^\infty f(x)\mu(x)\, dx$$

for all reasonable test functions f, then we will call μ the neighbor spacing density of the sequence X. For our matrix groups we will consider the eigenvalues as being ordered $\theta_1 \leq \theta_2 \leq \ldots$ and determine a function μ for which

$$\lim_{N\to\infty} \int_{U(N)} \frac{1}{N} \sum_{n=1}^{N-1} f(\tilde{\theta}_{n+1} - \tilde{\theta}_n) dU_N = \int_0^\infty f(x)\mu(x)\, dx$$

and similarly for R^+, R^-, P.

COMB. FORMULA FOR NGHBR SPAC.VIA CORRS.

For a sequence $X : \theta_1 \leq \cdots \leq \theta_N$ let

$$S_n(s, X) := \#\{j : \theta_{j+n} - \theta_j \leq s\},$$

and

$$C_m(s, X) := \#\{B \subset \{1, \ldots, N\} : |B| = m, \max_{j,k \in B} |\theta_j - \theta_k| \leq s\}$$

Lemma. For any X,

$$C_{m+2}(s, X) = \sum_{n \geq m} \binom{n}{m} S_{n+1}(s, X).$$

Proof. Take an $m+2$-tuple of indices $i_0 < i_1 < \cdots < i_{m+1}$ with endpoints $\theta_{i_{m+1}} - \theta_{i_0} \leq s$. Let $n = i_{m+1} - i_0$;the endpoints are counted in $S_n(s, X)$. There are $\binom{n-1}{m}$ sets of points of size m between these endpoints, which can be counted in $C_{m+2}(s, X)$. Therefore, $C_{m+2} = \sum \binom{n-1}{m} S_n$.

In general, the relation $a_m = \sum_{n \geq m} \binom{n}{m} b_n$ can be inverted to give

$$b_m = \sum_{n \geq m} (-1)^{n-m} \binom{n}{m} a_n.$$

Corollary.

$$S_{m+1}(s, X) = \sum_{n \geq m} (-1)^{n-m} \binom{n}{m} C_{n+2}(s, X)$$

or, after adjusting the indices,

$$S_m(s, X) = \sum_{n \geq m} (-1)^{n-m-1} \binom{n-2}{m-1} C_n(s, X).$$

Let $\tilde{X}_U$ be the sequence of normalized eigenangles of U.

We want to compute

$$\begin{aligned}\int_0^s \mu_1(x)\,dx : &= \text{Prob}\ \{\text{Neighboring eigenangles are} < s \text{ apart}\} \\ &= \lim_{N\to\infty} \frac{1}{N} \int_{U(N)} S_1(s, \tilde{X}_U)\,dU_N\end{aligned}$$

and more generally

$$\begin{aligned}\int_0^s \mu_m(x)\,dx : &= \text{Prob}\ \{m\text{th neighboring eigenangles are} < s \text{ apart}\} \\ &= \lim_{N\to\infty} \frac{1}{N} \int_{U(N)} S_m(s, \tilde{X}_U)\,dU_N.\end{aligned}$$

Applying the n-correlation calculation with

$$f(\theta_1, \ldots, \theta_n) = \prod_{1\le j<k\le n} \chi_{[0,s]}(|\theta_j - \theta_k|)$$

gives,

$$\lim_{N\to\infty} \frac{1}{N} \int_{U(N)} C_n(s, \tilde{X_U}) dU_N = \lim_{N\to\infty} \frac{1}{N} \int_{U(N)} \sum_{\substack{B\subset\{1,\ldots,N\}\\ |B|=n}} f(\theta_B) dU_N$$

$$= \frac{1}{(n-1)!} \int_{[0,s]^{n-1}} \det_{n\times n} S(x_j - x_k)\big|_{x_1=0} dx_2 \ldots dx_n$$

Thus, by our corollary,

$$\int_0^s \mu_m(x)\ dx = \sum_{n=m+1}^{\infty} \frac{(-1)^{n-m-1}}{(n-1)!} \binom{n-2}{m-1} \int_{[0,s]^{n-1}}$$

$$\det_{n\times n} S(x_j - x_k)\big|_{x_1=0} dx_2 \ldots dx_n$$

In particular, for the nearest neighbor spacing, we have

$$\mu_1(s) = \frac{d}{ds} \sum_{n=2}^{\infty} \frac{(-1)^n}{(n-1)!} \int_{[0,s]^{n-1}} \det_{n\times n} S(x_j - x_k)\big|_{x_1=0} dx_2 \ldots dx_n$$

Now, for any symmetric, even, translation invariant function g,

$$\frac{d}{ds}\int_{[0,s]^m} g(x_1,\ldots,x_m)dx_1\ldots dx_m = m\int_{[0,s]^{m-1}} g(s,x_2,\ldots,x_m)dx_2\ldots dx_m$$

Therefore,

$$\begin{aligned}\mu_1(s) &= \frac{d^2}{ds^2}\sum_{n=2}^{\infty}\frac{(-1)^n}{n!}\int_{[0,s]^n}\det_{n\times n} S(x_j-x_k)\ dx_1dx_2\ldots dx_n \\ &= \frac{d^2}{ds^2}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\int_{[0,s]^n}\det_{n\times n} S(x_j-x_k)\ dx_1dx_2\ldots dx_n.\end{aligned}$$

Also, temporarily letting

$$F(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!} \int_{[0,s]^n} \det_{n \times n} S(x_j - x_k)\, dx_1 dx_2 \ldots dx_n,$$

we have

$$\begin{aligned} \mu_m(s) &= \frac{d^2}{ds^2} \frac{d^{m-1}}{dz^{m-1}} \left(\frac{F(z) - 1 - z \int_0^s \det_{1\times 1} S\ d\theta}{z^2} \right) \Bigg|_{z=-1} \\ &= \frac{d^2}{ds^2} \frac{d^{m-1}}{dz^{m-1}} \frac{F(z)}{z^2} \Bigg|_{z=-1} \end{aligned}$$

GRAM'S IDENTITY

An identity of Gram is helpful for our further considerations. For an interval J and integrable functions ϕ_j and ψ_j,

$$
\begin{aligned}
&\det_{N\times N}\Big(I + z\int_J \phi_j(x)\psi_k(x)\ dx\Big) \\
&\qquad = \sum_{n=0}^{N} \frac{z^n}{n!} \int_{J^n} \Big(\det_{n\times n} \sum_{h=1}^{N} \phi_h(x_j)\psi_h(x_k)\Big)\ dx_1 \ldots dx_n .
\end{aligned}
$$

Proof . The left-hand-side of is

$$
\begin{aligned}
&\det_{N\times N}\left(I + z\int_J \phi_j(x)\psi_k(x)\,dx\right)\\
&\quad= \sum_{\sigma\in\pi_N} \operatorname{sgn}(\sigma) \prod_{j=1}^{N} \left(\delta_{j,\sigma j} + z\int_J \phi_j(x)\psi_{\sigma j}(x)\,dx\right)\\
&\quad= \sum_{\sigma\in\pi_N} \operatorname{sgn}(\sigma) \sum_{A\subset N} \prod_{j\notin U} \delta_{j,\sigma j} \prod_{j\in U} z\int_J \phi_j(x)\psi_{\sigma j}(x)\,dx\\
&\quad= \sum_{A\subset N} z^{|A|} \sum_{\sigma\in\pi_A} \operatorname{sgn}(\sigma) \prod_{j\in A} \int_J \phi_j(x)\psi_{\sigma j}(x)\,dx\\
&\quad= \sum_{n=0}^{N} z^n \sum_{\substack{A\subset N\\ |A|=n}} \det_A \int_J \phi_j(x)\psi_k(x)\,dx
\end{aligned}
$$

and the right-hand-side is

$$
\begin{aligned}
&= \sum_{n=0}^{N} \frac{z^n}{n!} \int_{J^n} \sum_{\sigma\in\pi_n} \operatorname{sgn}(\sigma) \prod_{j=1}^{n} \sum_{h=1}^{N} \phi_h(x_j)\psi_h(x_{\sigma j})\ dx_1 \ldots dx_N \\
&= \sum_{n=0}^{N} \frac{z^n}{n!} \int_{J^n} \sum_{\sigma\in\pi_n} \operatorname{sgn}(\sigma) \sum_{\lambda:[1,n]\to[1,N]} \prod_{j=1}^{n} \phi_{\lambda j}(x_j)\psi_{\lambda j}(x_{\sigma j})\ dx_1 \ldots dx_N \\
&= \sum_{n=0}^{N} \frac{z^n}{n!} \sum_{\sigma\in\pi_n} \operatorname{sgn}(\sigma) \sum_{\lambda:[1,n]\to[1,N]} \prod_{j=1}^{n} \int_J \phi_{\lambda j}(x)\psi_{\lambda\sigma^{-1}j}(x)\ dx \\
&= \sum_{n=0}^{N} \frac{z^n}{n!} \sum_{\lambda:[1,n]\to[1,N]} \sum_{\sigma\in\pi_n} \operatorname{sgn}(\sigma) \prod_{j=1}^{n} \int_J \phi_{\lambda j}(x)\psi_{\lambda\sigma^{-1}j}(x)\ dx \\
&= \sum_{n=0}^{N} \frac{z^n}{n!} \sum_{\lambda:[1,n]\to[1,N]} \det\left(\int_J \phi_{\lambda j}(x)\psi_{\lambda k}(x)\ dx \right)
\end{aligned}
$$

If λ is not 1-1, then this det is 0. If λ is 1-1, each image occurs $n!$

INTERVALS WITH EXACTLY n EIGENVALUES

$$E_{G(N)}(n, J) = \text{meas}\{A \in G(N) : A \text{ has } n \text{ eigs in } J\}$$

which have precisely n eigenvalues in the interval J. Let χ be the characteristic function of the interval J. First of all,

$$\sum_{n=0}^{N} (1+z)^n E_{G,N}(n, J) = \int_{G(N)} \prod_{j=1}^{N} (1 + z\chi(\theta_j)) dG_N$$

since for any $A \in G(N)$ with n eigenvalues in J, the integrand is $(1+z)^n$. Expanding gives

$$\int_{G(N)} \prod_{j=1}^{N} (1 + z\chi(\theta_j)) dG_N \quad = \quad \sum_{n=0}^{N} z^n \binom{N}{n} \int_{G(N)} \prod_{j=1}^{n} \chi(\theta_j) dG_N (1)$$

Next by Gaudin's Lemma,

$$\binom{N}{n}\int_{G(N)}\prod_{j=1}^{n}\chi(\theta_j)dG_N = \frac{z^n}{n!}\int_{J^n}\det_{n\times n} S_{G(N)}(\theta_j,\theta_k)\ d\theta_1\ldots d\theta_n$$

Thus,

$$\sum_{n=0}^{N} z^n\binom{N}{n}\int_{G(N)}\prod_{j=1}^{n}\chi(\theta_j)dG_N = \sum_{n=0}^{N}\frac{z^n}{n!}\int_{J^n}\det_{n\times n} S_{G(N)}(\theta_j,\theta_k)\ d\theta_1\ldots d\theta_n.$$

Now, for each $G(N)$ there are ϕ and ψ such that

$$S_{G(N)}(x,y) = \sum_{h=1}^{N}\phi_{h,G}(x)\psi_{h,G}(y)$$

$$\sum_{n=0}^{N}\frac{z^n}{n!}\int_{J^n}\det_{n\times n} S_{G(N)}(\theta_j,\theta_k)\ d\theta_1\ldots d\theta_n = \det_{N\times N}\left(I+z\int_J \phi_{j,G}(\theta)\psi_{k,G}(\theta)\ d\theta\right).$$

Let $M_{J,G(N)}$ denote the $N \times N$ matrix with entries

$$m_{j,k} = \int_J \phi_{j,G}(\theta)\psi_{k,G}(\theta)\ d\theta.$$

Then

$$\det_{N\times N}\left(I + z\int_J \phi_{j,G}(\theta)\psi_{k,G}(\theta)\ d\theta\right) = \prod_{j=1}^{N}\left(1 + z\lambda_{j,G(N)}(J)\right)$$

where the $\lambda_{j,G(N)}(J)$ are the eigenvalues of $M_{J,G(N)}$.

We claim that if the kernel is symmetric (i.e. $S_G(x,y) = S_G(y,x)$), then the eigenvalues of $M_{J,G(N)}$ are also the eigenvalues of the integral operator $K_{J,G(N)}$ defined by

$$(K_{J,G(N)}f)(\theta) \quad = \quad \int_J S_{G(N)}(\theta,\mu)f(\mu)\ d\mu$$

acting on the N-dimensional space generated by $\{\psi_j(x) : 1 \leq j \leq N\}$.

Proof. Suppose that λ is an eigenvalue of $M_{J,G(N)}$ corresponding to an eigenvector $\vec{v} = (b_1, \ldots, b_N)'$ where the prime indicates transpose. Then, for each j,

$$\lambda b_j = \sum_{k=1}^{N} m_{jk} b_k = \sum_{k=1}^{N} b_k \int_J \phi_j(\theta)\psi_k(\theta)\ d\theta$$

for each j. Multiplying both sides by $\psi_j(\mu)$ and summing over j, we obtain

$$\begin{aligned}
\lambda \sum_{j=1}^{N} b_j \psi_j(\mu) &= \int_J \Big(\sum_{j=1}^{N} \phi_j(\theta)\psi_j(\mu)\Big)\Big(\sum_{k=1}^{N} b_k\psi_k(\theta)\Big)\ d\theta \\
&= \int_J S_{G(N)}(\theta,\mu)\Big(\sum_{k=0}^{N} b_k\psi_k(\theta)\Big)\ d\theta \\
&= \int_J S_{G(N)}(\mu,\theta)\Big(\sum_{k=1}^{N} b_k\psi_k(\theta)\Big)\ d\theta = K_{J,G(N)} \sum_{k=1}^{N} b_k\psi_k(\mu)
\end{aligned}$$

so that λ is an eigenvalue of $K_{J,G(N)}$ corresponding to the eigenfunction $f(\mu) = \sum_{k=1}^{N} b_k \psi_k(\mu)$.

Recapitulating, we have found that

$$\begin{aligned}
\sum_{n=0}^{N} (1+z)^n E_{G,N}(n,J) &= \sum_{n=0}^{N} \frac{z^n}{n!} \int_{J^n} \det_{n\times n} S_{G(N)}(\theta_j,\theta_k)\, d\theta_1 \ldots d\theta_n \\
&= \prod_{j=1}^{N} \big(1 + z\lambda_{j,G(N)}(J)\big)
\end{aligned}$$

where the $\lambda_{j,G(N)}(J)$ are the eigenvalues of the integral operator $K_{J,G(N)}$ defined by

$$(K_{J,G(N)} f)(\theta) = \int_J S_{G(N)}(\theta,\mu) f(\mu)\, d\mu.$$

It can be shown that this equation scales appropriately for each G

so that the large N limit can be taken. This results in

$$\begin{aligned}\sum_{n=0}^{\infty}(1+z)^n E_G(n,J) &= \sum_{n=0}^{\infty}\frac{z^n}{n!}\int_{J^n}\det_{n\times n} S_G(\theta_j,\theta_k)\; d\theta_1\ldots d\theta_n \\ &= \prod_{j=1}^{\infty}\big(1+z\lambda_{j,G}(J)\big)\end{aligned}$$

where the $\lambda_{j,G}(J)$ are the eigenvalues of the integral operator $K_{J,G}$ defined by

$$(K_{J,G}f)(\theta) = \int_J S_G(\theta,\mu)f(\mu)\; d\mu.$$

The function $F(z)$ is equal to each of the above with G=U. Thus, we find for $\mu_m(s)$ that

$$\mu_m(s) = \frac{d^2}{ds^2}\frac{d^{m-1}}{dz^{m-1}}\left(z^{-2}\prod_{j=1}^{\infty}(1+z\lambda_{j,U}([0,s]))\right)\Bigg|_{z=-1}.$$

jTH LOWEST EIGENVALUE

Let

$$\nu_{G(N)}(j, s)$$

be the density function for the jth lowest eigenvalue so that

$$\begin{aligned} &\text{meas}\{A \in G(N) : \text{ the } j\text{th eigenvalue } \theta_j \text{ is smaller than } s\} \\ &= \int_0^s \nu_{G(N)}(j, x)\ dx. \end{aligned}$$

Then the set of $A \in G(N)$ with $\theta_j > s$ is the disjoint union of the set of A with exactly n eigenangles in $[0, s]$ for $n = 0, 1, \ldots, j - 1$. Thus,

$$\int_s^\infty \nu_{G(N)}(j, x) dx = \sum_{n=0}^{j-1} E_{G(N)}(n, [0, s]).$$

Therefore,

$$\nu_{G(N)}(j,s) = -\frac{d}{ds}\sum_{n=0}^{j-1}\frac{d^n}{dz^n}\prod_{j=1}^{N}(1+z\lambda_{G(N),j}([0,s]))\big|_{z=-1}$$

In the large N limit, this becomes

$$\nu_{G}(j,s) = -\frac{d}{ds}\sum_{n=0}^{j-1}\frac{d^n}{dz^n}\prod_{j=1}^{\infty}(1+z\lambda_{G,j}([0,s]))\big|_{z=-1}.$$

For example,

$$\nu_{G}(1,s) = -\frac{d}{ds}\prod_{j=1}^{\infty}(1-\lambda_{G,j}([0,s]).$$

RELATIONS BETWEEN EIGENVALUES

We show how the eigenvalues $\lambda_{j,G(N)}J)$ of the matrices $M_{J,G(N)}$ of (1) are related to each other. In the case that $J = [-s, s]$, note that if $\psi(\theta)$ is an eigenfunction of $M_{[-s,s],U(N)}$ with eigenvalue λ then $\psi(-\theta)$ is also an eigenfunction with eigenvalue λ, since

$$\lambda\psi(\theta) = \int_{-s}^{s} S_N(\theta - \mu)\psi(\mu)\, d\mu$$

implies that

$$\begin{aligned} \lambda\psi(-\theta) &= \int_{-s}^{s} S_N(-\theta - \mu)\psi(\mu)\, d\mu \\ &= \int_{-s}^{s} S_N(\theta + \mu)\psi(\mu)\, d\mu \\ &= \int_{-s}^{s} S_N(\theta - \mu)\psi(-\mu)\, d\mu \end{aligned}$$

Therefore, if $\psi(\theta) + \psi(-\theta) \neq 0$, then it is also an eigenfunction with eigenvalue λ. A similar comment holds for $\psi(\theta) - \psi(-\theta)$. Consequently, each eigenfunction can be taken to be even or odd. The even eigenfunctions are also eigenfunctions of the integral equation with kernel

$$\frac{S_N(\mu - \theta) + S_N(\mu + \theta)}{2}$$

and the odd eigenfunctions are also eigenfunctions of the integral equation with kernel

$$\frac{S_N(\mu - \theta) - S_N(\mu + \theta)}{2}.$$

In general, if a matrix B is a "checkerboard" matrix, then the determinant of b factors. Specifically, if $b_{j,k} = 0$ whenever $i + j$ is odd, then

$$\det_{N \times N}(b_{j,k}) = \det_{[(N+1)/2]}(b_{2i-1,2j-1}) \det_{[N/2] \times [N/2]}(b_{2i,2j})$$

where $[x]$ is the greatest integer less than or equal to x.

We have such a factorization for $\det(I - M_{[-s,s],U(N)})$. Using the fact that

$$\sum_h (\delta_{jh} - \cos(j\theta)\cos(h\theta))(\delta_{hk} - \sin(h\theta)\sin(k\theta)) = \delta_{jk} - \cos(k-j)\theta$$

we deduce from (1) (see also Mehta (10.2.6)) that

$$\det(I - M_{[-s,s],U(N)}) = \det(I - M_{[-s,s],O(N)})\det(I - M_{[-s,s],P(N)}).$$

This gives a factorization

$$\prod_{j=1}^{2N}(1 - \lambda_{j,U(2N)}(s)) = \prod_{j=1}^{N}(1 - \lambda_{j,O(N)}(s))(1 - \lambda_{j,P(N)}(s))$$

into even and odd eigenvalues. In particular, in the limit we have

$$\prod_{j=1}^{\infty}(1 - \lambda_{j,U}(s)) = \prod_{j=1}^{\infty}(1 - \lambda_{j,P}(s))(1 - \lambda_{j,O}(s)).$$

Alternatively, we have

$$\prod_{j=1}^{\infty}(1-\lambda_{j,P}(s)) = \prod_{j=1}^{\infty}(1-\lambda_{2j,U}(s))$$

and

$$\prod_{j=1}^{\infty}(1-\lambda_{j,R^+}(s)) = \prod_{j=1}^{\infty}(1-\lambda_{2j-1,U}(s))$$

provided that the $\lambda_{j,U}(s)$ are indexed so that an even index j corresponds to an even eigenfunction and an odd index j is for an odd eigenfunction of the integral operator with kernel $S_U(x,y) = S(x-y)$.

PROOF OF GAUDIN'S LEMMA

Let π_M be the symmetric group on $\{1, \ldots, M\}$. Then,

$$\det_{M \times M} \big(f(\theta_j, \theta_k) \big)$$

$$\sum_{\sigma \in \pi_M} \operatorname{sgn}(\sigma) \prod_{j=1}^{M} f(\theta_j, \theta_{\sigma j}).$$

If $\sigma M \neq M$, then

$$\int_J \prod_{j=1}^{M} f(\theta_j, \theta_{\sigma j}) d\theta_M$$

$$= \prod_{\substack{j=1 \\ \sigma j \neq M}}^{M-1} f(\theta_j, \theta_{\sigma j}) \int_J f(\theta_{\sigma^{-1} M}, \theta_M) f(\theta_M, \theta_{\sigma M})\, d\theta_M$$

$$= f(\theta_{\sigma^{-1} M}, \theta_{\sigma M}) \prod_{\substack{j=1 \\ \sigma j \neq M}}^{M-1} f(\theta_j, \theta_{\sigma j})$$

For a permutation $\sigma \in \pi_M$ with $\sigma M \neq M$ define a permutation $\sigma' \in \pi_{M-1}$ by

$$\sigma' j = \begin{cases} \sigma j & \text{if } \sigma j \neq M \\ \sigma M & \text{if } \sigma j = M \end{cases}$$

Then the above may be reexpressed as

$$\int_J \prod_{j=1}^{M} f(\theta_j, \theta_{\sigma j}) d\theta_M = C \prod_{j=1}^{M-1} f(\theta_j, \theta_{\sigma' j}).$$

Clearly, each permutation σ' arises from $(M-1)$ different σ. Note also that $\mathrm{sgn}(\sigma') = -\mathrm{sgn}(\sigma)$. Thus, we have

$$\begin{aligned}
&\int_J \sum_{\substack{\sigma \in \pi_M \\ \sigma M \neq M}} \mathrm{sgn}(\sigma) \prod_{j=1}^{M} f(\theta_j, \theta_{\sigma j}) d\theta_M \\
&\qquad = -(M-1)C \sum_{\sigma' \in \pi_{M-1}} \mathrm{sgn}(\sigma') \prod_{j=1}^{M-1} f(\theta_j, \theta_{\sigma' j}) \\
&\qquad = -(M-1)C \det_{M-1} \big(f(\theta_k, \theta_j) \big).
\end{aligned}$$

Now consider the σ for which $\sigma M = M$; now let σ' be defined by

$\sigma' j = \sigma j$ for $j \leq M - 1$. Then, for these σ, we have

$$\begin{aligned} &\int_J \prod_{j=1}^{M} f(\theta_j, \theta_{\sigma j}) d\theta_M \\ &\quad = \prod_{j=1}^{M-1} f(\theta_j, \theta_{\sigma j}) \int_J f(\theta_m, \theta_M)\ d\theta_M \\ &\quad = D \prod_{j=1}^{M-1} f(\theta_j, \theta_{\sigma' j}). \end{aligned}$$

These σ' have the same sign as the σ they came from. Therefore,

$$\begin{aligned}
\int_J \sum_{\substack{\sigma\in\pi_M \\ \sigma M = M}} \operatorname{sgn}(\sigma) \prod_{j=1}^{M} f(\theta_j, \theta_{\sigma j}) d\theta_M
&= D \sum_{\sigma'\in\pi_{M-1}} \operatorname{sgn}(\sigma') \prod_{j=1}^{M-1} f(\theta_j, \theta_{\sigma' j}) \\
&= D \det_{M-1} \big(f(\theta_k, \theta_j)\big).
\end{aligned}$$

Combining the two cases we obtain the Lemma.