

The Big Picture

Applications to physics and the real world

Jon Keating
Notes by Shahed Sharif

1 Introduction

Consider $\zeta(s)$. RH says that the zeros lie on the line $\Re(s) = 1/2$, or alternately if the zeros are $1/2 + it_n$, then t_n are all real. Assuming RH, what are the distribution of the zeros?

There is a connection between statistics of t_n and the statistics of eigenvalues of random matrices.

Random matrix theory has its roots in quantum mechanics.

2 Classical Mechanics

Consider a particle moving along a trajectory $\vec{r}(t)$. Newton's equations allow us to determine $\vec{r}(t)$. A fundamental concept in classical mechanics is *energy*, a function of \vec{r} and \vec{r}' , the position and velocity. In many situations, the energy $E(\vec{r}, \vec{r}')$ is a constant.

Now consider a complex system with many particles; for example, air molecules. Can one solve Newton's equations in this problem? Answer: no, too many particles, too many differential equations. To handle this, Maxwell, Boltzmann, et al. developed *statistical mechanics*. We make the assumption that the total energy of the particles is conserved, and that the probability of the position and velocity of each particle, subject to the energy constraint, is uniform, i.e. all configurations are equally likely. (These assumptions are known as *ergodicity*.) This formulation is much easier to solve, and also very accurate.

3 Chaos

Poincaré, but developed by Birkoff, Smale, et al.: Motion can be complex even in very simple systems, so complex that it is indistinguishable from a

random process. For example, one gas particle in a box, or geodesic motion on compact surfaces of negative curvature (Hadamard).

But one can structure dynamics around periodic orbits. Periodic orbits form a set of measure zero, but give structure to the space of all possible orbits. By analogy, consider the rationals inside the reals. For example, periodic orbits are dense in the space of all orbits.

4 Quantum Mechanics

Every particle has a *wave function* $\psi(s, t)$ with $|\psi|^2 =$ the probability density for position s at time t . The wave function satisfies Schrödinger's equation. For example, for a particle in a box, the equation is

$$\nabla^2\psi = -k^2\psi \tag{1}$$

subject to $\psi = 0$ on the boundary. The Schrödinger equation involves a self-adjoint operator.

ψ lives in some Hilbert space, so can be written in terms of some orthonormal basis. Then Schrödinger equation becomes a matrix equation on the coefficients (Heisenberg).

In quantum mechanics, energy is discrete, not continuous. In fact, the energy values are given by the eigenvalues of the matrix.

One can solve very simple problems with this framework, e.g. hydrogen atom or ideal spring. What about complex quantum mechanical problems, e.g. the nucleus of a (heavy!) atom?

Wigner, 1950s: create an analogue for statistical mechanics. The idea is to let the entries of the matrix be random variables. One constraint is that the matrix should be symmetric or Hermitian. Then one can do statistics, e.g. distribution of eigenvalues.

Dyson, Mehta, Gaudin, 1960s.

Dyson's three-fold way: if the only structure in the Hilbert space is that there is a Hermitian inner product, then there are only three "ensembles" of random matrices, GUE ($\beta = 2$), GOE ($\beta = 1$), and GSE ($\beta = 4$). (Gaussian Unitary/Orthogonal/Symplectic Ensemble)

5 Disordered systems

Suppose one has a wire with an electron moving through it. There will be impurities in the wire, and the electron will scatter off the impurities. The relationship between current and voltage is affected by the impurities. For very small wires, say inside a microprocessor, and at low temperatures, one needs to use quantum mechanics. Wigner's approach: look at the statistics over the space of all possible wires, so let the impurities be randomly distributed in the wire.

If one averages over random "force", get "supersymmetric" integrals. When we evaluate these, get random matrix theory.

6 Quantum Chaos

The path of the electron in the wire, for example, exhibits chaotic behavior under quantum mechanics.

Bohigas, Giannoni, Schmit: if the motion is chaotic, then the energy levels should behave like the eigenvalues of random matrices in "generic" cases. Open problem: in what situations does this hold? Part of the problem is there is no space to average over.

Another approach: the trace formula (Gutzwiller). Periodic states can be related to energy levels in the high energy limit ($k \rightarrow \infty$ in the Schrödinger equation (1)). This says that the statistics of energy levels corresponds to the statistics of periodic orbits. (Berry, Bogomolny-K, Sieber, Richter)

Berry: periodic orbits obey an analogue of the prime number theorem.

Pollicott answered questions about periodic orbits using techniques from analytic number theory.

7 Quantum Field Theory

Motivated by attempt to marry quantum mechanics and relativity. No connections with number theory yet, but many with random matrices.

QCD — quantum chromodynamics. Feynmann diagrams are related to random matrices.

Quantum gravity. Solutions in general relativity are smooth manifolds. In quantum mechanics, want to average over perturbations of the manifold.

This is too hard. However, if we triangulate the manifold and consider discrete perturbations, one gets a random matrix model.

String theory gives random matrix models.

Consider Dyson's 3-fold way. In QCD, *chiral symmetries* give us 7 other random matrix ensembles. Zimbauer: the 10 ensembles correspond to the large symmetric spaces. This classification also appears (Katz-Sarnak) in the theory of L -functions: there are 10 symmetry spaces of L -functions.

8 Speculation

Is there a direct connection between quantum mechanics and number theory, not via random matrix theory? Perhaps one can show RH by showing that the t_n are energy levels of some quantum system. Maybe the associated quantum system is chaotic, which is why random matrix theory comes up as the "bridge".

Explicit formulas: what's going on with the coincidences from the trace formula?

Perhaps statistical properties of periodic orbits are the same as the statistical properties of the primes.