Nearest Neighbour Spacing

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Given a sequence of numbers with mean spacing of one:

0.4274265847095507	15.11801754076215	29.13146285981258
1.498046222018098	16.46829723490161	29.75595368422236
2.294731862788573	16.95839238329879	31.26865039390550
3.948051950911685	18.65484724195002	31.90196392915219
4.791923474650027	18.94492439824802	33.07465967390136
5.338164026156317	19.94010173399443	34.28657870441268
6.142064919962254	21.03703089368745	35.05629175528733
7.045415878713115	22.06376183543616	36.72238402472082
8.367321946374252	23.55399096931206	37.16506226976983
9.288369536214912	24.00928783449144	38.42185901187239
10.39951439966856	25.29662380802344	39.06642285856689
10.68043705136611	26.64747707499640	40.54638091185787
11.21348203345075	27.51815198379607	40.93832483449469
13.35047594595835	28.39949893146165	42.16328590411256
14.15217925465181		

we plot the nearest neighbour spacing by plotting the (normalised) histogram of the spacing distance between consecutive numbers.

1.07062	0.965838	0.731964
0.796686	1.35028	0.624491
1.65332	0.491396	1.5127
0.843872	1.69515	0.633314
0.546241	0.290077	1.1727
0.803901	0.995177	1.21192
0.903351	1.09693	0.769713
1.32191	1.02673	1.66609
0.921048	1.49023	0.442678
1.11114	0.455297	1.2568
0.280923	1.28734	0.644564
0.533045	1.35085	1.47996
2.13699	0.870675	0.391944
0.801703	0.881347	1.22496

We might wish to check that the sequence of numbers above has a distribution that agrees with that of eigenvalues from matrices selected randomly from $U(N)$, the group of $N \times N$ unitary matrices, with respect to Haar measure. To do this, we want to be able to calculate the nearest neighbour spacing distribution $p_N(s)$ for random $U(N)$ matrices in order to compare with the results of plotting the spacings listed in the table above. Note that $p_N(s)ds$ is therefore the probability that a given eigenvalue of a random $N \times N$ unitary matrix is followed by a space of length between s and $s + ds$ between it and the next eigenvalue. There are several ways to calculate the $U(N)$ nearest neighbour spacing distribution, and it is instructive to know more than one because the same techniques can be applied to calculating other random matrix statistics. We will concentrate on the unitary group in this lecture, but the methods presented here extend almost immediately to other random matrix ensembles. Most of this material is to be found (in more generality, but with some deviations of notation) in "Spacing distributions in random matrix ensembles" by Forrester and section 9 of "Notes on eigenvalue distributions for the classical compact groups" by Conrey, both to be found in *Recent Perspectives in Random Matrix Theory and Number Theory*.

1 The Wigner Surmise

Historically, it is interesting to note that Wigner hypothesised a surprisingly accurate form of the nearest neighbour spacing distribution for matrix size going to infinity. Wigner was actually considering complex Hermitian matrices with random Gaussian entries, the so-called Gaussian Unitary Ensemble. We now know that in the large-matrix limit local statistics of this ensemble agree with those of the random unitary matrices we are interested in. Wigner identified the quadratic repulsion of the eigenvalues of these matrices, and guessed the form

$$p(s) = c_1 s^2 e^{-c_2 s^2} \tag{1}$$

for the nearest neighbour spacing distribution, determining the constants by the normalization requirements

$$\int_0^\infty p(s)ds = 1 \quad \text{and} \quad \int_0^\infty s\, p(s)ds = 1. \tag{2}$$

This lead to

$$p(s) = \frac{32s^2}{\pi^2} e^{-4s^2/\pi}, \tag{3}$$

which is a remarkably accurate approximation to the true distribution.

The nearest neighbour distribution is not an easy statistic to calculate - certainly not as easy as a given correlation function. However, since it is a local statistic, the nearest neighbour spacing distribution can be calculated from a knowledge of all the n-point correlation functions. We will demonstrate this, but first will introduce a related statistic, the gap probability.

2 Intervals with n eigenvalues

Let $E_N(n, J)$ be the probability that a random matrix (with respect to Haar measure) from $U(N)$ has exactly n eigenvalues in the interval J. For example, the interval J might be $[0, s]$

for some $0 < s \leq 2\pi$. Of particular interest is $E_N(0,[0,s])$, because this "gap probability" is related to the nearest neighbour spacing, $p_N(s)$.

$E_N(0,[0,s])$ is the probability that the interval $[0,s]$ contains no eigenvalues for a random $N \times N$ unitary matrix. If we add a tiny increment of length on to the original interval, giving us a length $s+\delta s$, then there are two possibilities: either the extra segment δs contains at least one eigenvalue or it contains none. If the latter is true, then the entire interval $[0, s+\delta s]$ is empty, and the probability of this is $E_N(0,[0,s+\delta s])$. Thus $E_N(0,[0,s]) - E_N(0,[0,s+\delta s])$ is the probability that the interval $[0,s]$ is empty but δs is not. If we take the limit as $\delta s \to 0$, then the probability that δs contains two or more eigenvalues will be swamped by the probability that there is just one eigenvalue in δs. Thus $-\frac{dE_N(0,[0,s])}{ds}\delta s$ is the probability that an empty interval $[0,s]$ is followed by one eigenvalue in the interval δs. Alternatively, $-\frac{dE_N(0,[0,s])}{ds}$ tells us how likely it is that an eigenvalue will be followed by an empty gap of length at least s. Repeating the argument above we see that

$$\left(\frac{-\frac{dE_N(0,[0,s])}{ds} + \frac{dE_N(0,[0,s+\delta s])}{ds}}{\delta s}\right)\delta s = \frac{d^2 E_N(0,[0,s])}{ds^2}\delta s \tag{4}$$

tells us how likely it is that two consecutive eigenvalues will be separated by a distance between s and $s+\delta s$. As we have used the notation $p_N(s)ds$ for just such a probability, we see that

$$C_N p_N(s) = \frac{d^2 E_N(0,[0,s])}{ds^2}, \tag{5}$$

where C_N is a normalisation constant.

For a different derivation of this same relationship, see the second half of section 9.1 "Notes on Eigenvalue Distributions for the Classical Compact Groups" by Conrey. He uses the notation $\mu_1(s) = \lim_{N\to\infty} \frac{2\pi}{N} p_N(\frac{2\pi s}{N})$ and his $F(z)$ evaluated at $z=-1$ is $\lim_{N\to\infty} E_N(0,[0,2\pi s/N])$, and so for $m=1$, equation (9.4) in Conrey's notes is the same as (5) above, but in the $N \to \infty$ limit. In fact, Conrey's $F(z)$ is the $N\to\infty$ limit of the generating function $\mathcal{E}_N(-z,J)$ that we will introduce in the next section, with the interval $J=[0,2\pi s/N]$.

3 Nearest neighbour spacing in terms of correlation functions

We have already set $E_N(n,J)$ to be the probability that a random matrix (with respect to Haar measure) from $U(N)$ has exactly n eigenvalues in the interval J. Here $\chi_J(\theta)$ is the characteristic function that takes the value 1 if θ lies in the interval J and takes the value 0 if θ does not lie in J. For an interval of fixed length, say s, we can express the $E_N(n,J)$ statistics in terms of the correlation functions R_k, N. Recall the definition of the correlation functions:

$$R_{k,N}(\theta_1,\ldots,\theta_k) = \frac{N!}{(N-k)!}\int_0^{2\pi}\cdots\int_0^{2\pi} P(\theta_1,\cdots,\theta_N)d\theta_{k+1}\cdots d\theta_N, \tag{6}$$

where $P(\theta_1, \cdots, \theta_N)$ is the joint probability density function of the eigenvalues. As we are only interested in random matrices from $U(N)$ in this lecture, for now

$$P(\theta_1, \cdots, \theta_N) = \frac{1}{(2\pi)^N N!} \prod_{1 \le j < k \le N} |e^{i\theta_k} - e^{i\theta_j}|^2. \tag{7}$$

We define a generating function for the $E_N(n, J)$ statistics:

$$\begin{aligned}
\mathcal{E}_N(z, J) &= \sum_{n=0}^{N} (1-z)^n E_N(n, J) \\
&= \int_{U(N)} \prod_{j=1}^{N} (1 - z\chi_J(\theta_j)) dX \\
&= \frac{1}{(2\pi)^N N!} \int_0^{2\pi} \cdots \int_0^{2\pi} \prod_{j=1}^{N} (1 - z\chi_J(\theta_j)) \prod_{1 \le j < k \le N} |e^{i\theta_k} - e^{i\theta_j}|^2 d\theta_1 \cdots d\theta_N \\
&= 1 + \sum_{k=1}^{N} \frac{(-z)^k}{k!} \int_J \cdots \int_J R_{k,N}(\theta_1, \ldots, \theta_k) d\theta_1 \cdots d\theta_k. \qquad (8)
\end{aligned}$$

So we see that we can express the generating function for the $E_N(n, J)$ probabilities in terms of the correlation functions. In particular, $\mathcal{E}_N(1, [0, s]) = E_N(0, [0, s])$ and so the nearest neighbour spacing distribution is

$$C_N p_N(s) = \frac{d^2}{ds^2} \mathcal{E}_N(1, [0, s]). \tag{9}$$

4 Fredholm determinant method

It is all very well in theory to write the nearest neighbour spacing distribution in terms of the correlation functions, but it is not straightforward to evaluate $p_N(s)$ from (8). One method to do so uses Fredholm determinants and integral equations and will be detailed briefly here.

Recall that we have a determinantal form for the correlation functions (see equation (6.4) of Conrey's "Notes on eigenvalue distributions for the classical compact groups", or for more general treatment, section 4 of Fyodorov's "Introduction to the random matrix theory: Gaussian Unitary Ensemble and beyond", both in *Recent Perspectives...*):

$$R_{k,N}(\theta_1, \ldots, \theta_k) = \det_{k \times k} [S_N(\theta_j - \theta_\ell)], \tag{10}$$

with

$$S_N(\theta) = \frac{1}{2\pi} \sum_{n=1}^{N} e^{i(n - N/2 - 1/2)\theta} = \frac{1}{2\pi} \frac{\sin(N\theta/2)}{\sin(\theta/2)}. \tag{11}$$

Note that there is a difference of a factor of 2π between this definition and that in Conrey's "Notes on eigenvalue distributions for the classical compact groups" in *Recent Perspectives in Random Matrix Theory and Number Theory*. Both have been used extensively, and this can cause some confusion.

Thus

$$S_N(\theta_j - \theta_\ell) = \frac{1}{2\pi} \sum_{h=1}^{N} e^{i(h-N/2-1/2)\theta_j} e^{-i(h-N/2-1/2)\theta_\ell}. \tag{12}$$

Now we need Gram's Identity

$$\sum_{k=0}^{N} \frac{z^k}{k!} \int_{J^k} \det_{k\times k} \left[\sum_{h=1}^{N} \phi_h(\theta_j)\psi_h(\theta_\ell) \right] \, d\theta_1 \dots d\theta_k = \det_{N\times N} \Big[I + z \int_J \phi_j(\theta)\psi_\ell(\theta) \, d\theta \Big]. \tag{13}$$

(For a straightforward proof of this identity see Lemma 5, Section 9.2 of Conrey's "Notes on eigenvalue distributions for the classical compact groups" in *Recent Perspectives....*)

We plug (12) into (10) and substitute the whole expression into (8) to arrive at

$$\mathcal{E}_N(z, J) = 1 + \sum_{k=1}^{\infty} \frac{(-z)^k}{k!} \int_J \cdots \int_J \det_{k\times k} \Big[\frac{1}{2\pi} \sum_{h=1}^{N} e^{i(h-N/2-1/2)\theta_j} e^{-i(h-N/2-1/2)\theta_\ell} \Big] d\theta_1 \cdots d\theta_k. \tag{14}$$

With $\phi_h(\theta_j) = e^{i(h-N/2-1/2)\theta_j}$ and $\psi_h(\theta_\ell) = e^{-i(h-N/2-1/2)\theta_\ell}$, we apply Gram's Identity to achieve

$$\mathcal{E}_N(z, J) = \det_{N\times N} \Big[I - \frac{z}{2\pi} \int_J e^{i(j-k)\theta} \, d\theta \Big]. \tag{15}$$

For finite N this determinant can be computed for a numerical evaluation of

$$C_N p_N(s) = \frac{d^2}{ds^2} \mathcal{E}_N(1, [0, s]). \tag{16}$$

Or, equivalently, if $\lambda_{j,N}(J)$ are the eigenvalues of the matrix with entries

$$m_{jk} = \frac{1}{2\pi} \int_J e^{i(j-k)\theta} d\theta, \tag{17}$$

then

$$\mathcal{E}_N(z, J) = \prod_{j=1}^{N} (1 - z\lambda_{j,N}(J)). \tag{18}$$

These are also the eigenvalues of the integral equation

$$\lambda_{j,N}(J) f_j(x) = \int_J S_N(x, y) f_j(y) dy, \tag{19}$$

with eigenfunctions $f_j(x)$. For a proof of this see the bottom of page 140 in Conrey's "Notes on eigenvalue distributions for the classical compact groups".

We often want the limit of the nearest neighbour spacing distribution as the matrix size N goes to infinity. Taking the limit of $\mathcal{E}_N(z, J)$ only makes sense if the interval J shrinks by a factor $1/N$ as N grows; that is, the interval length remains constant when measured in units of the mean spacing between the eigenvalues, $\frac{2\pi}{N}$. Thus we want the limit

$$\mathcal{E}(z, (-s/2, s/2)) = \lim_{N\to\infty} \mathcal{E}_N(z, [-\tfrac{s\pi}{N}, \tfrac{s\pi}{N}]). \tag{20}$$

(Note that shifting the interval from $[0, s]$ to $[-s/2, s/2]$ makes no difference as the correlation functions are translation invariant, and this way we will arrive at the traditional form of the answer.) This limit is hard to take in a determinantal expression like (15), but easy to adapt (19). It is convenient to redefine the eigenfunction as follows $h_j(t) = g_j(\frac{s}{2}t) = f_j(\frac{\pi st}{N})$ during the course of the calculation.

From (19):

$$\lambda_{j,N}([-s\pi/N, s\pi/N])f_j(\tfrac{2\pi t}{N}) = \int_{-s\pi/N}^{s\pi/N} \frac{1}{2\pi} \frac{\sin(N(\frac{2\pi t}{N} - y)/2)}{\sin((\frac{2\pi t}{N} - y)/2)} f_j(y)dy \tag{21}$$

$$\lambda_{j,N}([-s\pi/N, s\pi/N])g_j(t) = \int_{-s/2}^{s/2} \frac{\sin(N(\frac{2\pi t}{N} - \frac{2\pi y}{N})/2)}{\sin((\frac{2\pi t}{N} - \frac{2\pi y}{N})/2)} g_j(y)\frac{dy}{N}.$$

So in the limit, with $\lambda_j(s) = \lim_{N\to\infty} \lambda_{j,N}([-s\pi/N, s\pi/N])$, we have

$$\lambda_j(s)h_j(t) = \int_{-1}^{1} \frac{\sin(\pi(t-y)s/2)}{\pi(t-y)} h_j(y)dy. \tag{22}$$

This integral equation can be solved numerically, and then

$$p(s) = \frac{d^2}{ds^2} \prod_{j=1}^{\infty} (1 - \lambda_j(s)). \tag{23}$$

5 Painlevé method

One final method of numerically calculating the nearest neighbour spacing distribution is to relate the statistic to the solution of a nonlinear differential equation. Many statistics can be computed by this means, see for example Sections 3 and 4 of Forrester's "Spacing distributions in random matrix ensembles", in *Recent Perspectives in Random Matrix Theory and Number theory.* These differential equations are called Painlevé equations, having been classified by Painlevé.

Here we will just mention the fact that

$$p(s) = \frac{\pi^2}{3} s^2 \exp \int_0^{2\pi s} v(t)\frac{dt}{t}, \tag{24}$$

where v satisfies the nonlinear equation

$$(sv'')^2 + (v - sv')(v - sv' + 4 - 4(v')^2) - 16(v')2 = 0, \tag{25}$$

with the boundary condition

$$v(s) \sim_{s\to 0} -\frac{1}{15}s^2. \tag{26}$$

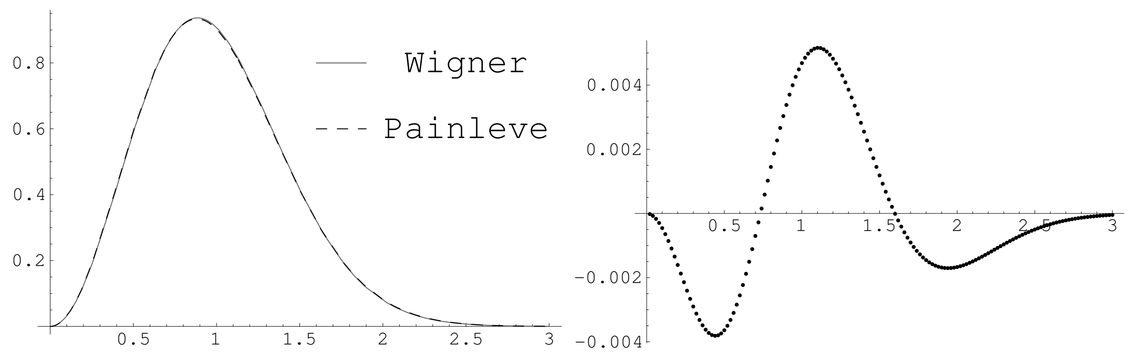

Figure 1: In the figure on the left we plot the Wigner surmise, see (3), against the large N limiting distribution for the nearest neighbour distribution calculated using the Painlevé method, using the first 200 terms in the series expansion of the solution. In the figure on the right we plot the difference between these two curves, showing how accurate Wigner's guess was.