Mean values of characteristic polynomials

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1 The unitary group: $U(N)$

The study of mean values of the Riemann zeta function,

$$\frac{1}{T}\int_0^T |\zeta(1/2+it)|^s dt, \tag{1}$$

can be elucidated by the examination of the mean values of random matrix characteristic polynomials [4] (see also the article of Keating "L-functions and the characteristic polynomials of random matrices" in *Recent Perspectives in Random Matrix Theory and Number Theory*). These latter mean values follow from a straightforward random matrix calculation which is detailed below.

We write the characteristic polynomial of A, an $N \times N$ unitary matrix, as

$$\Lambda_A(s) = \det(I - A^* s) = \prod_{n=1}^{N} \left(1 - s e^{-i\theta_n}\right), \tag{2}$$

where the $e^{i\theta_n}$ are the eigenvalues of A and A^* is the conjugate transpose of A.

We want to mimic mean values of the Riemann zeta function by calculating moments of $|\Lambda_A(e^{i\theta})|$. Since this is a function of the eigenvalues, an average of $|\Lambda_A(e^{i\theta})|^s$ can be performed over the group $U(N)$ with respect to Haar measure using the Weyl integration formula:

$$\int_{U(N)} |\Lambda_A(e^{i\theta})|^s dA_{Haar} \tag{3}$$
$$= \frac{1}{(2\pi)^N N!} \int_0^{2\pi} \cdots \int_0^{2\pi} \prod_{1\le j<m\le N} \left|e^{i\theta_j} - e^{i\theta_m}\right|^2 \left|\prod_{n=1}^{N} (1 - e^{i(\theta-\theta_n)})\right|^s d\theta_1 \cdots d\theta_N.$$

Integrals similar to this (those which contain a product with the structure $\prod_{1\le j<m\le N} |x_j - x_m|$) can be dealt with by the use of a form of Selberg's integral (described at length in Chapter 17 of [5]). The integral which we need for (3) is

$$\begin{aligned} &J(a,b,\alpha,\beta,\gamma,N) \\ &= \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\left|\prod_{1\le j<\ell\le N}(x_j-x_\ell)\right|^{2\gamma}\times\prod_{j=1}^{N}(a+ix_j)^{-\alpha}(b-ix_j)^{-\beta}dx_j \\ &= \frac{(2\pi)^N}{(a+b)^{(\alpha+\beta)N-\gamma N(N-1)-N}}\cdot\prod_{j=0}^{N-1}\frac{\Gamma(1+\gamma+j\gamma)\Gamma(\alpha+\beta-(N+j-1)\gamma-1)}{\Gamma(1+\gamma)\Gamma(\alpha-j\gamma)\Gamma(\beta-j\gamma)}. \end{aligned} \tag{4}$$

In the above formula, a, b, α, β and γ are complex numbers, $\mathrm{Re}\,a$, $\mathrm{Re}\,b$, $\mathrm{Re}\,\alpha$ and $\mathrm{Re}\,\beta$ are all greater than zero, $\mathrm{Re}\,(\alpha+\beta)>1$ and

$$-\frac{1}{N}<\mathrm{Re}\,\gamma<\min\left(\frac{\mathrm{Re}\,\alpha}{N-1},\frac{\mathrm{Re}\,\beta}{N-1},\frac{\mathrm{Re}\,(\alpha+\beta-1)}{2(N-1)}\right). \tag{5}$$

In attempting to coerce (3) into the form of Selberg's integral, we note that

$$\left|e^{i\theta_j}-e^{i\theta_m}\right| = 2\left|\sin(\theta_j/2-\theta_m/2)\right|, \tag{6a}$$

and similarly

$$\left|1-e^{i(\theta-\theta_p)}\right| = 2\left|\sin(\theta_p/2-\theta/2)\right|. \tag{6b}$$

Therefore we can write (3) as

$$\begin{aligned} \int_{U(N)}|\Lambda_A(e^{i\theta})|^s dA_{Haar} &= \frac{2^{N(N-1)}2^{sN}}{N!(2\pi)^N} \\ \times\int_0^{2\pi}\cdots\int_0^{2\pi}&\prod_{1\le j<m\le N}|\sin(\theta_j/2-\theta_m/2)|^2\prod_{n=1}^{N}|\sin(\theta_n/2-\theta/2)|^s\,d\theta_1\cdots d\theta_N, \end{aligned} \tag{7}$$

and we note that this integral is in fact independent of θ, which we eliminate to obtain

$$\begin{aligned} \int_{U(N)}|\Lambda_A(1)|^s dA_{Haar} &= \frac{2^{N(N-1)}2^{sN}2^N}{N!(2\pi)^N}\int_0^{\pi}\cdots\int_0^{\pi}d\theta_1\cdots d\theta_N\prod_{1\le j<m\le N}|\sin(\theta_j-\theta_m)|^2 \\ &\quad\times\prod_{n=1}^{N}|\sin\theta_n|^s. \end{aligned} \tag{8}$$

If we write

$$\begin{aligned} \sin(\theta_j-\theta_m) &= \sin\theta_j\cos\theta_m-\cos\theta_j\sin\theta_m \\ \frac{\sin(\theta_j-\theta_m)}{\sin\theta_j\sin\theta_m} &= \cot\theta_m-\cot\theta_j, \end{aligned} \tag{9}$$

then the moment can be written as

$$\int_{U(N)} |\Lambda_A(1)|^s dA_{Haar} = \frac{2^{N^2+sN}}{N!(2\pi)^N} \int_0^\pi \cdots \int_0^\pi d\theta_1 \cdots d\theta_N \prod_{1\le j<m\le N} |\cot\theta_m - \cot\theta_j|^2$$
$$\times \prod_{n=1}^N \left(\sin^2\theta_n\right)^{N-1} \prod_{n=1}^N |\sin\theta_n|^s . \quad (10)$$

A change of variables, $x_n = \cot\theta_n$, gives us

$$\int_{U(N)} |\Lambda_A(1)|^s dA_{Haar} = \frac{2^{N^2+sN}}{N!(2\pi)^N} \int_{-\infty}^\infty \cdots \int_{-\infty}^\infty dx_1 \cdots dx_N \prod_{1\le j<m\le N} |x_m - x_j|^2$$
$$\times \prod_{n=1}^N ((1+ix_n)(1-ix_n))^{-N-s/2}$$
$$= \frac{2^{N^2+sN}}{N!(2\pi)^N} J(1,1,N+s/2,N+s/2,1,N)$$
$$= \prod_{j=1}^N \frac{\Gamma(j)\Gamma(s+j)}{(\Gamma(j+s/2))^2} \equiv M_N(s), \quad (11)$$

where J is the form of Selberg's integral defined in (4). Considering the conditions, listed in (5), on the various parameters, (11) holds if $-1 < \mathrm{Re} s$.

As N becomes large, we can look at the asymptotic form of $M_N(s)$ and we see that for an integer $k \le N-1$,

$$\int_{U(N)} |\Lambda_A(1)|^{2k} dA_{Haar} = \prod_{j=1}^N \frac{(j-1)!(2k+j-1)!}{(j+k-1)!^2}$$
$$= \frac{0!1!\cdots(N-1)!(2k)!(2k+1)!\cdots(2k+N-1)!}{k!(k+1)!\cdots(k+N-1)!k!(k+1)!\cdots(k+N-1)!}$$
$$= \frac{0!1!\cdots(k-1)!(k+N)!(k+N+1)!\cdots(2k+N-1)!}{N!(N+1)!\cdots(k+N-1)!k!(k+1)!\cdots(2k-1)!}$$
$$= \frac{0!1!\cdots(k-1)!}{k!(k+1)!\cdots(2k-1)!}[(N+1)\cdots(k+N)]\,[(N+2)\cdots (k+N+1)]\cdots\cdots[(k+N)\cdots(2k+N-1)]$$
$$= \prod_{j=0}^{k-1} \frac{j!}{(k+j)!} N^{k^2} + O(N^{k^2-1}). \quad (12)$$

For non-integer exponent the asymptotics are also known and can be written as

$$M_N(s) \sim \frac{(G(1+s/2))^2}{G(1+s)} N^{(s/2)^2}, \quad (13)$$

where $G(z)$ is the Barnes G-function [2, 8, 7] defined by

$$G(1+z) = (2\pi)^{z/2} e^{-[(1+\gamma)z^2+z]/2} \prod_{n=1}^\infty \left[(1+z/n)^n e^{-z+z^2/(2n)}\right]. \quad (14)$$

This function has zeros at the negative integers, $-n$, with multiplicity n ($n = 1, 2, 3\ldots$), and is related to the gamma function, as can be seen by

$$\begin{aligned} G(1) &= 1, \\ G(z+1) &= \Gamma(z)\, G(z). \end{aligned} \tag{15}$$

Having determined the mean values of $|\Lambda_A(1)|^s$ in (11), it is straightforward to obtain both the integer moments and the value distribution of the logarithm of $|\Lambda|$. These log moments are generated by $M_N(s)$, so we have

$$M_N(s) = \sum_{j=0}^{\infty} \frac{m_j}{j!} s^j, \tag{16}$$

where $m_j = \int_{U(N)} (\log |\Lambda_A(1)|)^j dA_{Haar}$. Also, defining Q_j as the cumulants of $\log |\Lambda|$,

$$\log M_N(s) = \sum_{j=1}^{\infty} \frac{Q_j}{j!} s^j. \tag{17}$$

Thus

$$\begin{aligned} Q_n &= \frac{d^n}{ds^n} \log M_N(s)\big|_{s=0} \\ &= \frac{d^{n-1}}{ds^{n-1}} \sum_{j=1}^{N} \left(\psi(j+s) - \psi(j+s/2)\right)\big|_{s=0} \qquad (18) \\ &= \sum_{j=1}^{N} \left(\psi^{(n-1)}(j+s) - \frac{1}{2^{n-1}} \psi^{(n-1)}(j+s/2)\big|_{s=0}\right) \\ &= \frac{2^{n-1}-1}{2^{n-1}} \sum_{j=1}^{N} \psi^{n-1}(j), \qquad (19) \end{aligned}$$

where we have introduced the polygamma functions $\psi^{(n)}(z) = \frac{d^{n+1}}{dz^{n+1}} \log \Gamma(z)$.

If we consider the cumulants when N is large then we can employ the asymptotic expansion for the polygamma functions [1]

$$\psi^{(n)}(z) \sim (-1)^{n-1} \left[\frac{(n-1)!}{z^n} + \frac{n!}{2z^{n+1}} + \sum_{k=1}^{\infty} B_{2k} \frac{(2k+n-1)!}{(2k)! z^{2k+n}} \right] \tag{20}$$

for $|z| \to \infty$ with $|\arg z| < \pi$. Here the B_{2k} are Bernoulli numbers.

From (20) we see that the second cumulant is of order $\log N$ as N becomes large:

$$\begin{aligned} Q_2 &= \frac{1}{2} \sum_{j=1}^{N} \psi^{(1)}(j) \\ &= \frac{1}{2} \sum_{j=1}^{N} \left(\frac{1}{j} + \frac{1}{2j^2} + O\left(\frac{1}{j^3}\right) \right) \\ &\sim \frac{1}{2} \log N. \end{aligned} \tag{21}$$

An identical argument shows that the leading order term of all cumulants Q_n with $n \geq 3$ is a constant, and it is clear that $Q_1 = 0$.

This knowledge of the cumulants is alone enough to show that the value distribution of $(\mathrm{Re}\log\Lambda)/\sqrt{Q_2}$ $(= \log|\Lambda|/\sqrt{Q_2})$ is Gaussian in the limit $N \to \infty$.

Let $P(x)$ be the probability density for values of $|\Lambda_A(1)|$. That is, $P(x)dx$ is the probability that $|\Lambda|$ takes a value between x and $x + dx$. Then it is a standard result from probability that $P(x)$ is related to the moment $M_N(s)$ calculated earlier by the transform

$$P(x) = \frac{1}{2\pi i x} \int_{c-i\infty}^{c+i\infty} M_N(s) x^{-s} ds, \tag{22}$$

for some $c > 0$. A change of variables then shows us that $\rho(x)$, the probability density for values of $(\mathrm{Re}\log\Lambda)/\sqrt{Q_2}$ is

$$\rho(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} M_N(s/\sqrt{Q_2}) e^{-xs} ds. \tag{23}$$

Writing this in terms of the cumulants:

$$\rho(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp(-isx - s^2/2 - iQ_3 s^3/(3!Q_2^{3/2}) + Q_4 s^4/(4!Q_2^2) + \cdots) ds. \tag{24}$$

Since

$$\frac{Q_j}{Q_2^{j/2}} \to 0 \quad \text{as} \quad N \to \infty, \quad \text{for} \quad j \geq 3, \tag{25}$$

as $N \to \infty$,

$$\begin{aligned} \rho(x) &\sim \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-iyx} e^{-y^2/2} dy \\ &\sim \frac{1}{\sqrt{2\pi}} \exp\left(\frac{-x^2}{2}\right). \end{aligned} \tag{26}$$

Although the limiting distribution is Gaussian, the convergence to the limit is slow. It is a theorem of Selberg's that the distribution of the real part of the logarithm of the Riemann zeta function also tends to a Gaussian for very large heights up the critical line, but if we plot this distribution for values of $\mathrm{Re}\log\zeta(1/2+it)$ with t around 10^{19}, we see it is far from having converged. Equating the density of Riemann zeros around that height with the density of the eigenvalues on the unit circle of the unitary matrices we are considering leads to the equivalence $N \sim \log t/2\pi$. Figure 1 shows that although at these values of N and t neither the Riemann distribution or the random matrix one have converged to the Gaussian, for $N = 42$ and $t \sim 10^{19}$, the distribution of $\log|\Lambda_A(1)|$ is a very good model for that of $\mathrm{Re}\log\zeta(1/2 + it)$.

The imaginary part of the logarithm can be considered similarly.

2 The Unitary Symplectic Group: $USp(2N)$

We are interested here in the group of symplectic unitary matrices, $USp(2N)$. These are $2N \times 2N$ matrices, A, with $AA^\dagger = 1$ and $A^t J A = J$, where $J = \begin{pmatrix} 0 & I_N \\ -I_N & 0 \end{pmatrix}$ and I_N

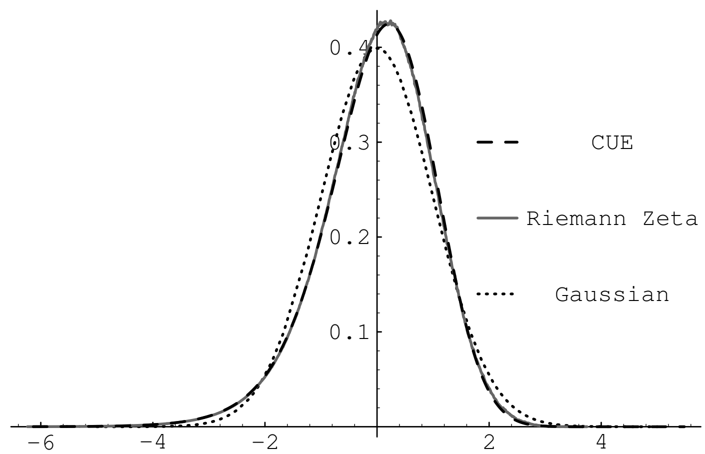

Figure 1: The value distribution for $\mathrm{Re}\log\Lambda$ for matrices from $U(42)$ (also called the CUE), Odlyzko's data for the value distribution of $\mathrm{Re}\log\zeta(1/2+it)$ near the 10^{20}th zero (taken from [6]), and the standard Gaussian, all scaled to have unit variance. (Figure from [4].)

is the $N \times N$ identity matrix. For these matrices, the eigenvalues lie on the unit circle and come in complex conjugate pairs. Thus the characteristic polynomial related to such a matrix with eigenvalues $e^{i\theta_1}, e^{-i\theta_1}, e^{i\theta_2}, e^{-i\theta_2}, \ldots e^{i\theta_N}, e^{-i\theta_N}$ takes the form

$$\Lambda_A(e^{i\theta}) = \prod_{n=1}^{N} \left(1 - e^{i(\theta-\theta_n)}\right)\left(1 - e^{i(\theta+\theta_n)}\right) \tag{27}$$

and

$$\Lambda_A(1) = 2^N \prod_{n=1}^{N} (1 - \cos\theta_n). \tag{28}$$

The moment of this characteristic polynomial at the point $e^{i\theta} = 1$ with respect to Haar measure on $USp(2N)$ is

$$\begin{aligned} \int_{USp(2N)} |\Lambda_A(1)|^s dA_{Haar} &= \frac{2^{N^2}}{\pi^N N!} 2^{Ns} \int_0^\pi \cdots \int_0^\pi d\theta_1 \cdots d\theta_N \prod_{1\le i<j\le N} (\cos\theta_j - \cos\theta_i)^2 \\ &\quad \times \prod_{k=1}^{N} \sin^2\theta_k \times \prod_{n=1}^{N} (1-\cos\theta_n)^s, \end{aligned} \tag{29}$$

This, after the transformation $x_j = \cos\theta_j$, becomes

$$\begin{aligned} \int_{USp(2N)} |\Lambda_A(1)|^s dA_{Haar} &= \frac{2^{N^2}}{\pi^N N!} 2^{Ns} \int_{-1}^{1} \cdots \int_{-1}^{1} dx_1 \cdots dx_N \prod_{1\le i<j\le N} (x_j - x_i)^2 \\ &\quad \times \prod_{k=1}^{N} (1-x_k)^{1/2+s}(1+x_k)^{1/2}. \end{aligned} \tag{30}$$

There is a form of Selberg's integral (detailed in [5]) which states that

$$\begin{aligned} &\int_{-1}^{1} \cdots \int_{-1}^{1} \prod_{1\le j<l\le n} |(x_j - x_l)|^{2\gamma} \prod_{j=1}^{n} (1-x_j)^{\alpha-1}(1+x_j)^{\beta-1} dx_j \\ &= 2^{\gamma n(n-1)+n(\alpha+\beta-1)} \prod_{j=0}^{n-1} \frac{\Gamma(1+\gamma+j\gamma)\Gamma(\alpha+j\gamma)\Gamma(\beta+j\gamma)}{\Gamma(1+\gamma)\Gamma(\alpha+\beta+\gamma(n+j-1))}, \end{aligned} \tag{31}$$

if $\mathrm{Re}\alpha > 0$, $\mathrm{Re}\beta > 0$ and $\mathrm{Re}\gamma > -\min\left(\frac{1}{n}, \frac{\mathrm{Re}\alpha}{n-1}, \frac{\mathrm{Re}\beta}{n-1}\right)$.

In our case $\gamma = 1$, $\alpha = 3/2 + s$ and $\beta = 3/2$, so

$$\begin{aligned} \int_{USp(2N)} |\Lambda_A(1)|^s dA_{Haar} &= \frac{2^{N^2}}{\pi^N N!} 2^{Ns} 2^{N^2+N+Ns} \prod_{j=0}^{N-1} \frac{\Gamma(2+j)\Gamma(3/2+s+j)\Gamma(3/2+j)}{\Gamma(2)\Gamma(3+s+N+j-1)} \\ &= 2^{2Ns} \prod_{j=1}^{N} \frac{\Gamma(1+N+j)\Gamma(1/2+s+j)}{\Gamma(1/2+j)\Gamma(1+s+N+j)} \\ &\equiv M_{Sp}(N,s). \end{aligned} \tag{32}$$

By studying the cumulants c_j in the expansion

$$M_{Sp}(N,s) = e^{c_1 s + c_2 s^2/2 + c_3 s^3/3! + c_4 s^4/4! + \cdots}, \tag{33}$$

as in the unitary case, we can examine the distribution of the characteristic polynomial and its logarithm (see [3] for more details) and compare these with the corresponding distributions for L-function values at the critical point in families showing symplectic symmetry. It is also useful to note that for large N the moment has the form

$$M_{Sp}(N,s) \sim 2^{s^2/2} \times \frac{G(1+s)\sqrt{\Gamma(1+s)}}{\sqrt{G(1+2s)\Gamma(1+2s)}} N^{s(s+1)/2}, \tag{34}$$

which for integer moments simplifies to

$$M_{Sp}(N,n) \sim \left(\prod_{j=1}^{n} (2j-1)!! \right)^{-1} N^{n(n+1)/2}. \tag{35}$$

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