Riemann, in his 1859 paper, connected the problem of determining $\pi(x)$, the number of primes $\leq x$, to the zeros of

$$\zeta(s) = \sum_1^\infty \frac{1}{n^s} = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1}, \quad \Re s > 1$$

David's talk:

$\zeta(s)$ extends to a meromorphic function with its only pole at $s=1$ (simple, residue 1)

Recall from complex analysis, to study zeros of an analytic function $f(z)$, one should look at $\frac{f'}{f}(z)$.

On the other hand, logarithmic derivative of a product is a sum. when applied to $\zeta(s)$, this allows us, with some modification to count primes.

$$-\frac{\zeta'}{\zeta}(s) = \sum_{1}^{\infty} \frac{\Lambda(n)}{n^s}, \quad \mathrm{Re}\, s > 1$$

where

$$\Lambda(n) = \begin{cases} \log p & \text{if } n = p^k \\ 0 & \text{otherwise} \end{cases}$$

Riemann's Trick

(he actually applied it to $\log \zeta(s)$ instead of $-\frac{\zeta'}{\zeta}(s)$, but in his notes one finds it also for $-\frac{\zeta'}{\zeta}$, which is usually attributed to Von Mangoldt).

Let $c > 0$. Then

$$\frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{y^s}{s}\, ds = \begin{cases} 1 & \text{if } y > 1 \\ 1/2 & y = 1 \\ 0 & 0 < y < 1 \end{cases}$$

(Cauchy principal value for the integral if $y=1$)

Thus, if $c > 1$

$$\sum_{n \leq x} \Lambda(n) = \frac{-1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{\zeta'}{\zeta}(s)\, \frac{x^s}{s}\, ds.$$

pulls out terms with $\frac{x}{n} \geq 1$, ie $n \leq x$

(take $\frac{\Lambda(n)}{2}$ if $x \in \mathbb{Z}$)

No harm in looking at $\psi(x) = \sum_{n \le x} \Lambda(n)$

instead of $\pi(x) = \sum_{p \le x} 1$

$$\psi(x) = \sum_{n \le x} \Lambda(n) = \sum_{p^k \le x} \log p = \underbrace{\sum_{p \le x} \log p}_{\substack{\text{call this} \\ \vartheta(x)}} + O(x^{1/2})$$

Relation between $\pi(x)$ and $\vartheta(x)$ via summation by parts:

$$\pi(x) = \sum_{p \le x} \log p \cdot \frac{1}{\log p} = \vartheta(x) \frac{1}{\log x} + \int_2^x \vartheta(t) \frac{dt}{t(\log t)^2}$$

so if $\vartheta(x) \sim x$ then $\pi(x) \sim \frac{x}{\log x}$

Conversely

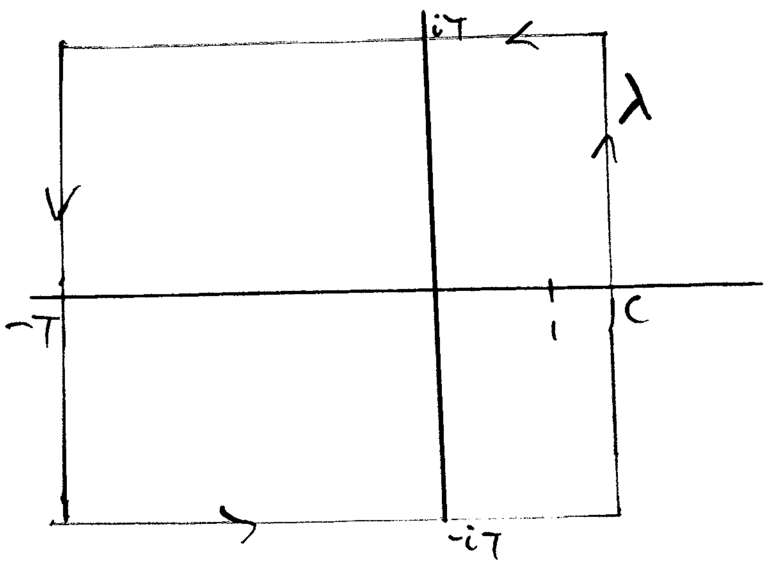
$$\vartheta(x) = \sum_{p \le x} 1 \cdot \log p = \pi(x) \log x - \int_2^x \pi(t) \frac{dt}{t}$$

if $\pi(x) \sim \frac{x}{\log x}$ then $\vartheta(x) \sim x$.

Hence PNT is equivalent to $\psi(x) \sim x$

Explicit formula for $\psi(x)$

$$\psi(x) = \frac{-1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{\zeta'}{\zeta}(s)\frac{x^s}{s}\,ds = \lim_{T\to\infty} \frac{-1}{2\pi i}\int_{\lambda} \frac{\zeta'}{\zeta}(s)\frac{x^s}{s}\,ds$$



Poles of $-\frac{\zeta'}{\zeta}(s)\frac{x^s}{s}$: 1) $s=0$, residue $-\frac{\zeta'}{\zeta}(0) = -\log 2\pi$

2) $s=1$, residue x (main contribution)

3) zeros of $\zeta(s)$. Trivial zeros $s=-2,-4,-6,\ldots$ (via functional eqn)
residues $\sum_{1}^{\infty} \frac{1}{2m} x^{-2m} = -\frac{1}{2}\log(1-x^{-2})$

· non-trivial zeros ρ, residues $-\sum_{\rho} \frac{x^\rho}{\rho}$

Explicit formula.

Let $x>1$. Then

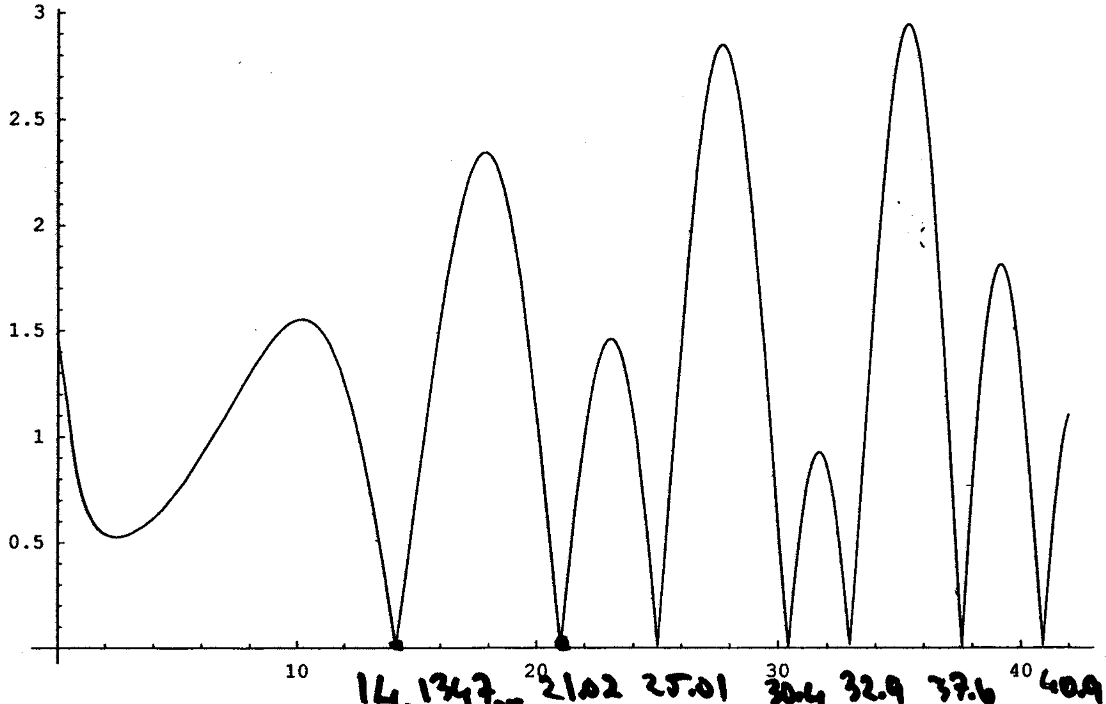
$$\psi(x) = x - \sum_{\rho} \frac{x^{\rho}}{\rho} - \log(2\pi) - \frac{1}{2}\log(1-x^{-2})$$

This shows there are infinitely many non-trivial zeros (otherwise r.h.s would be continuous in x, but l.h.s is a step function).

RH: $\rho = \frac{1}{2} + i\gamma$, $\gamma \in \mathbb{R}$

RH $\Leftrightarrow$ $\psi(x) = x + O(x^{1/2}(\log x)^2)$

In[2]:= `Plot[Abs[Zeta[1 / 2 + I * t]], {t, 0, 42}]`

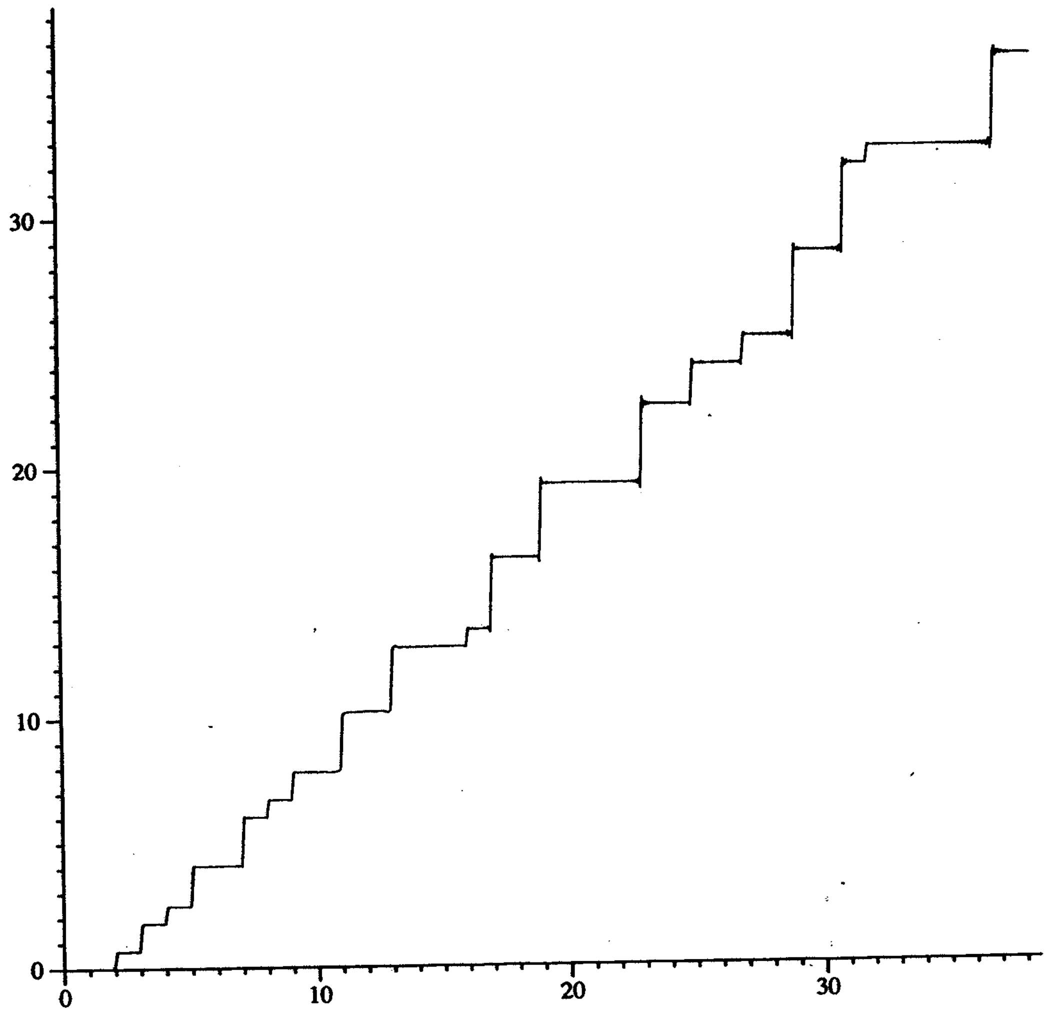


Out[2]= - Graphics -

γ

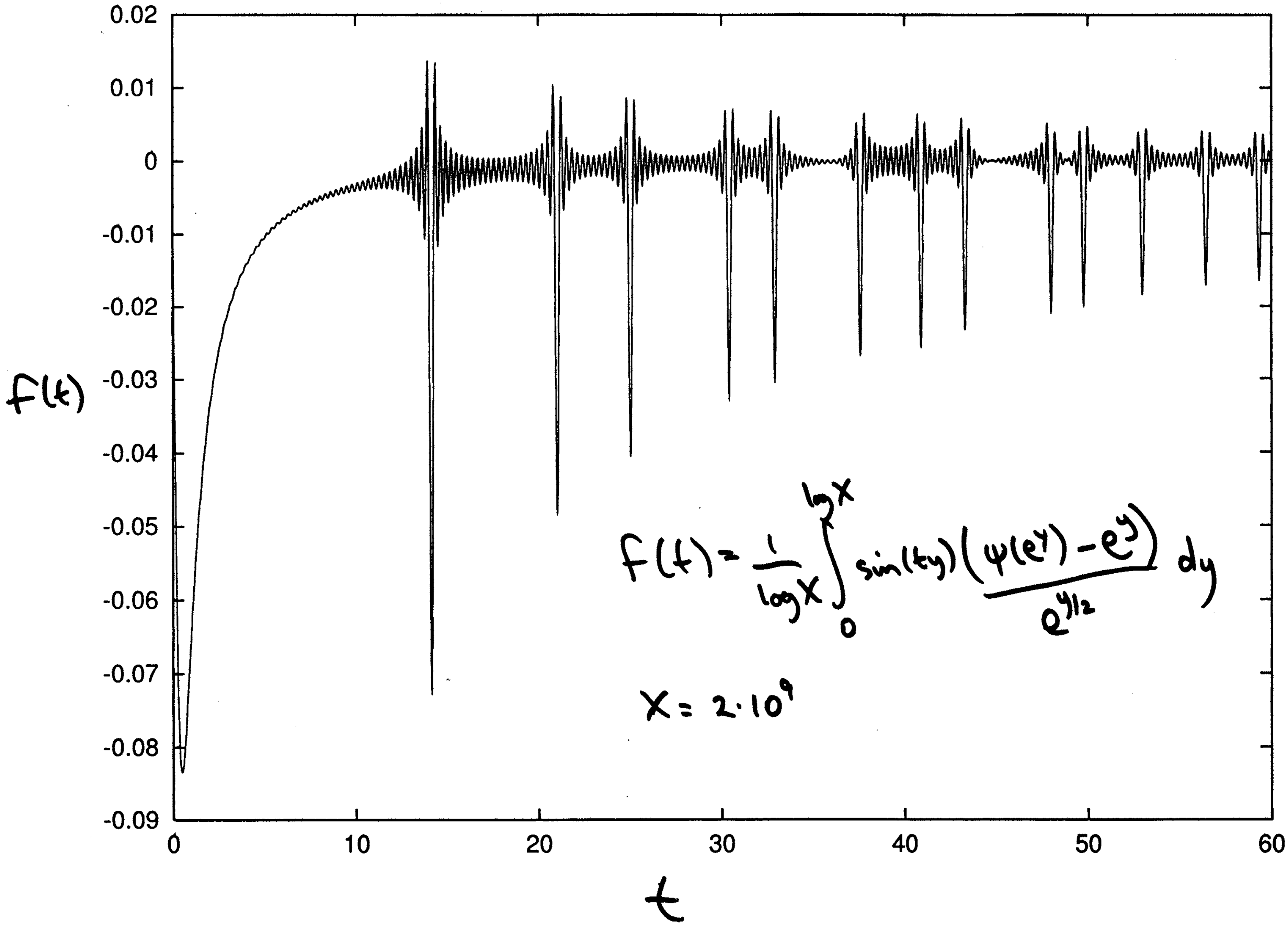
14.134725142
21.022039639
25.010857580
30.424876126
32.935061588
37.586178159
40.918719012
43.327073281
48.005150881
49.773832478
52.970321478
56.446247697
59.347044003
60.831778525
65.112544048
67.079810529
69.546401711
72.067157674
75.704690699
77.144840069
79.337375020

$\zeta(\frac{1}{2} \pm i\gamma) = 0$



Explicit formula for psi(x)

$$x - \sum_{j=1}^{1000} 2\,\mathrm{Re}\left(\frac{x^{\rho_j}}{\rho_j}\right) - \log(2\pi) - \frac{1}{2}\log(1 - x^{-2})$$


0.02
0.01
0
-0.01
-0.02
-0.03
-0.04
-0.05
-0.06
-0.07
-0.08
-0.09
0
10
20
30
40
50
60
f(t)
t
$f(t) = \frac{1}{\log X} \int_0^{\log X} \sin(ty) \left(\frac{\psi(e^y) - e^y}{e^{y/2}} \right) dy$
$X = 2 \cdot 10^9$

$$\Pi(x) = \pi(x) + \tfrac{1}{2}\pi(x^{\frac{1}{2}}) + \tfrac{1}{3}\pi(x^{\frac{1}{3}}) + \cdots$$

$$= \sum_{p^k \le x} \frac{1}{k} \qquad \left(\text{take } \tfrac{1}{k} \text{ if } x \text{ is a prime power}\right)$$

$$\pi(x) = \sum_{1}^{\infty} \frac{\mu(n)}{n}\, \Pi(x^{\frac{1}{n}})$$

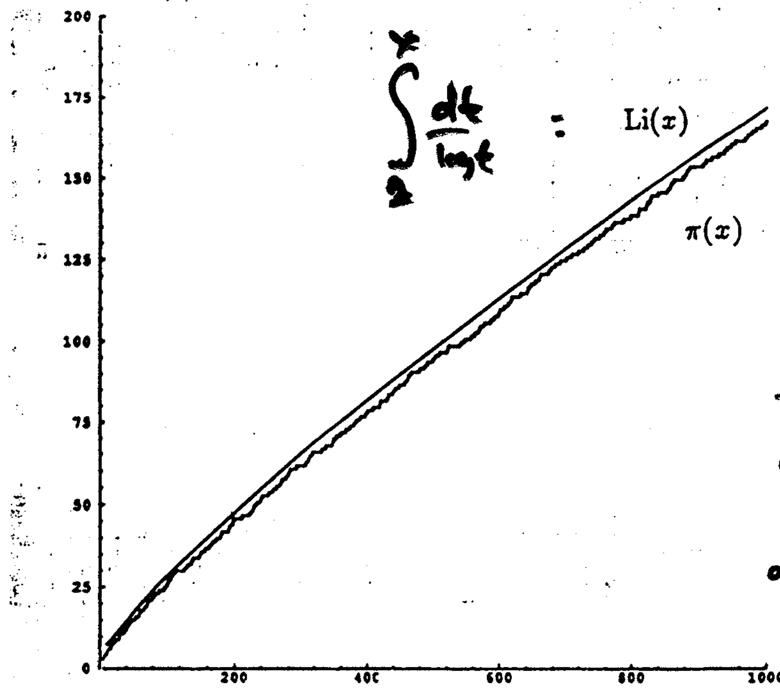
$$= \Pi(x) - \tfrac{1}{2}\Pi(x^{\frac{1}{2}}) - \tfrac{1}{3}\Pi(x^{\frac{1}{3}}) - \tfrac{1}{5}\Pi(x^{\frac{1}{5}}) + \tfrac{1}{6}\Pi(x^{\frac{1}{6}}) \ldots$$

Riemann's paper (proven by von Mangoldt)

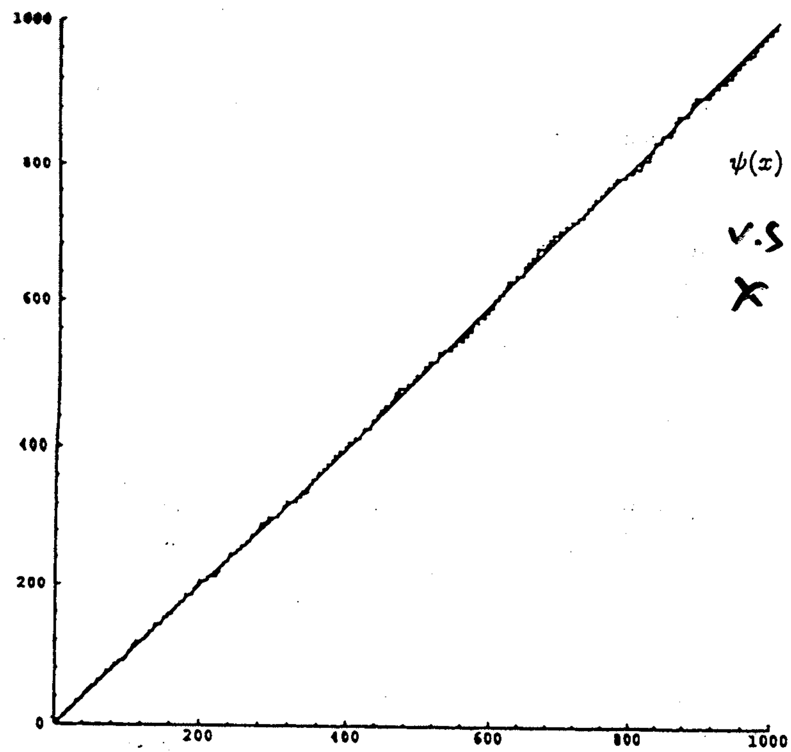
$x > 1$

$$\boxed{\Pi(x) = \mathrm{li}(x) - \sum_{\rho} \mathrm{li}(x^{\rho}) + \int_x^{\infty} \frac{du}{(u^2-1)\,u \log u} - \log 2}$$

$$\mathrm{li}(x) = \lim_{\eta \to 0^+} \int_0^{1-\eta} + \int_{1+\eta}^{x} \frac{du}{\log u}$$



$\pi(x) < Li(x)$
99.999973%
of the time, on a logarithmic scale.



$\psi(x) = \sum_{p^k \leq x} \log p$
v.s
x

Prime number theorem

$\pi(x) \sim \frac{x}{\log x}$ $\longleftrightarrow$ $\zeta(s) \neq 0$ when $\operatorname{Re} s = 1$

(proven 1896 by Hadamard, de la Vallée Poussin)

$$\pi(x) = \int_2^x \frac{dt}{\log t} + O(x^{\frac{1}{2}} \log x)$$

$\longleftrightarrow$

$$\psi(x) = x + O(x^{\frac{1}{2}} \log^2 x)$$

$\longleftrightarrow$

R.H.

The number of zeros of $\zeta(s)$

We'll, as before, apply Cauchy's thm and functional eqn to obtain an asymptotic formula for

$$N(T) = \left| \left\{ \rho = \beta + i\gamma \;\middle|\; \zeta(\rho) = 0,\ \begin{matrix} 0<\beta<1 \\ 0<\gamma\leq T \end{matrix} \right\} \right|$$

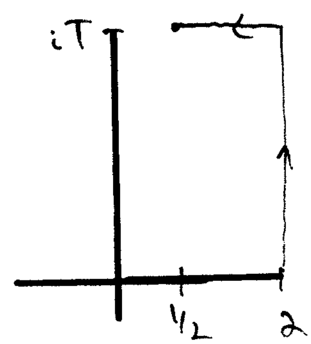
i.e number of non-trivial zeros in the critical strip above real axis, with imaginary part $\leq T$.

<u>Thm</u>

$$N(T) = \frac{T}{2\pi}\log\frac{T}{2\pi e} + \frac{7}{8} + S(T) + O(1/T)$$

where

$$S(T) = \frac{1}{\pi}\arg\zeta(1/2 + iT)$$



pf

Let $f(s)$ be analytic inside/on λ, a simple closed curve, non zero on λ.

Recall, number of zeros of $f(s)$ inside λ:

$$\frac{1}{2\pi i}\int_\lambda \frac{f'}{f}(s)\,ds = \frac{1}{2\pi i}\Delta_\lambda \log f(s)$$
$$\frac{1}{2\pi}\Delta_\lambda \arg f(s)$$

Rather than apply to $\zeta(s)$, better to use

$$\pi^{-s/2}\,\Gamma(s/2)\,\zeta(s)$$

as we'll be applying functional eqn. Also better to multiply by $s(s-1)$ to kill poles at $s=1$ of $\zeta(s)$, $s=0$ of $\Gamma(s/2)$

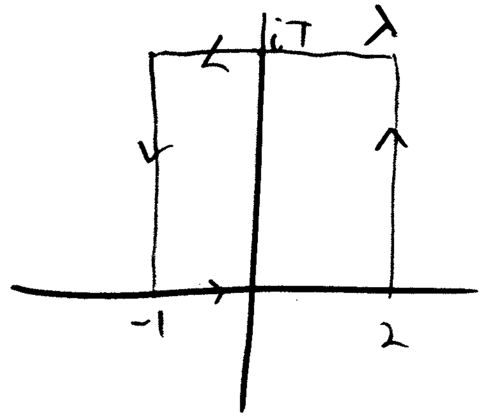
For simplicity, throw in an extra factor of $1/2$.

Let

$$\xi(s) = (s-1)\pi^{-\frac{s}{2}} \overbrace{\Gamma\left(\tfrac{s}{2}+1\right)}^{\frac{s}{2}\Gamma\left(\frac{s}{2}\right)} \zeta(s)$$

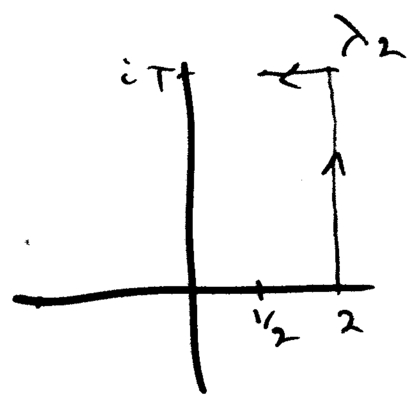
$$\xi(s) = \xi(1-s)$$

$$N(T) = \frac{1}{2\pi} \Delta_\lambda \arg \xi(s)$$



$$\xi(1-\sigma+it) = \overline{\xi(\sigma+it)}$$

$$\frac{1}{2\pi} \Delta_\lambda \arg \xi(s) = \frac{1}{\pi} \Delta_{\lambda_2} \arg \xi(s)$$



If f is a product then $\frac{f'}{f}$ is a sum of individual pieces.

Now

$$\Delta_{\lambda_2} \arg(s-1) = \frac{\pi}{2} + O(1/T)$$

$$\Delta_{\lambda_2} \arg \pi^{-s/2} = -\frac{T}{2}\log \pi$$

$$\Delta_{\lambda_2} \arg \Gamma(s/2+1) = \frac{1}{2}T\log\frac{T}{2e} + \frac{3}{8}\pi + O(1/T)$$

via Stirling's formula

Thus

$$N(T) = \frac{T}{2\pi}\log\frac{T}{2\pi e} + \frac{7}{8} + S(T) + O(1/T)$$

What's known about S(T)

$$S(T) = O(\log T) \quad \text{unconditionally}$$
$$O(\log T / \log\log T) \quad \text{on RH}$$

Littlewood

$$\frac{1}{T}\int_0^T S(t)\,dt = O\left(\log T/T\right),$$ i.e. on average lots of cancellation

Turing, explicitly (useful for checking numerically if all zeros up to height T have been found)

$$\left|\int_{t_1}^{t_2} S(t)\,dt\right| \leq 2.3 + .128\log(t_2/\pi)$$

for all $t_2 > t_1 > 168\pi$

Weil's Explicit formula

Let $\phi : \mathbb{R} \to \mathbb{R}$

$$\hat{\phi}(x) = \int_{-\infty}^{\infty} \phi(t) e^{-2\pi i t x}\, dt$$

Assume that $\hat{\phi}$ is smooth, and is compactly supported. Hence ϕ extends to an entire function function and is rapidly decreasing along real axis.

write $\rho = \frac{1}{2} + i\gamma$, non-trivial zeros of ζ.

Then

$$\sum \phi(\gamma) = \phi\left(\frac{i}{2}\right) + \phi\left(-\frac{i}{2}\right)$$

$$+ \frac{1}{2\pi} \int_{-\infty}^{\infty} \phi(t) \left(-\log \pi + \mathrm{Re} \frac{\Gamma'}{\Gamma}\left(\frac{1}{4} + it\right) \right) dt$$

$$- \frac{1}{2\pi} \sum_{1}^{\infty} \frac{\Lambda(n)}{\sqrt{n}} \left(\hat{\phi}\left(\frac{\log n}{2\pi}\right) + \hat{\phi}\left(-\frac{\log n}{2\pi}\right) \right)$$

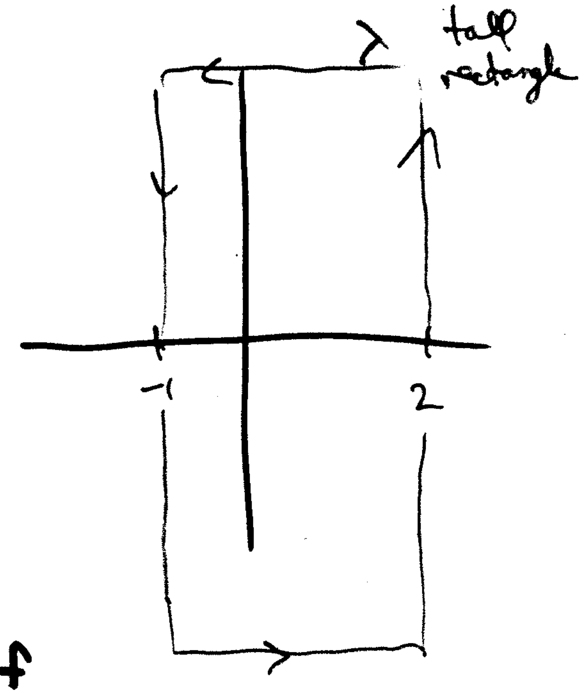
sum over all γ, counted with multiplicity. conjecture: all zeros are simple.

pf/

overloading notation

Let $\Lambda(s) = \pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s)$ and consider

$$\frac{1}{2\pi i}\int_{\lambda} \frac{\Lambda'}{\Lambda}(s)\,\phi\left(\frac{s-\frac{1}{2}}{i}\right)ds$$



Poles of integrand:

1) non-trivial zeros of $\zeta(s)$, residue $\phi(\gamma)\cdot$ multiplicity of the zero

2) $s=1$ (pole of $\zeta(s)$), residue $-\phi(-i/2)$

3) $s=0$ (pole of $\Gamma(s/2)$), residue $-\phi(i/2)$

On the other hand

$$\frac{\Lambda'}{\Lambda}(s) = -\frac{\Lambda'}{\Lambda}(1-s)$$

So our contour integral is, in the limit (a little bit of care needed on horizontal pieces.)

$$\frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty} \frac{\Lambda'}{\Lambda}(s)\left(\phi\left(\frac{s-1/2}{i}\right)+\phi\left(\frac{1/2-s}{i}\right)\right)ds$$

But

$$\frac{\Lambda'}{\Lambda}(s) = \underbrace{\frac{1}{2}\frac{\Gamma'}{\Gamma}(s/2) - \frac{\log\pi}{2}}_{\substack{\text{this contributes...}\\ \text{to the integral (shift line to Re } s = 1/2)}} + \overbrace{\frac{\mathcal{I}'}{\mathcal{I}}(s)}^{\text{next pg}}$$

$$\ldots \quad \frac{1}{2\pi}\int_{-\infty}^{\infty} \phi(t)\left(-\log\pi + \frac{1}{2}\frac{\Gamma'}{\Gamma}\left(1/4 + it/2\right) + \frac{1}{2}\frac{\Gamma'}{\Gamma}\left(1/4 - it/2\right)\right)dt$$

$$\frac{\zeta'}{\zeta}(s) = -\sum_{1}^{\infty} \frac{\Lambda(n)}{n^s}, \quad \mathbb{R}e\, s > 1$$

But $\displaystyle -\frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty} \phi\left(\frac{s-\frac{1}{2}}{i}\right) n^{-s}\, ds$

$$= -\frac{n^{-1/2}}{2\pi}\int_{-\infty}^{\infty} \phi(t)\, n^{-it}\, dt$$

$$= -\frac{n^{-1/2}}{2\pi}\,\hat{\phi}\left(\frac{\log n}{2\pi}\right)$$

Similarly for $\phi\left(\frac{\frac{1}{2}-s}{i}\right)$.

Montgomery (1973) was the 1st person to study the vertical distribution of the zeros.

Write a typical non trivial zero ρ of ζ as

$$\rho = \frac{1}{2} + i\gamma, \qquad (\text{RH} \Leftrightarrow \gamma\text{'s are real})$$

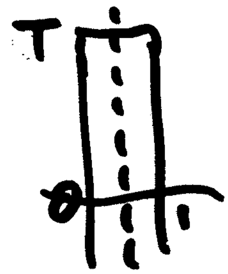
Assume RH and order the zeros

$$0 < \gamma_1 \leq \gamma_2 \leq \gamma_3 \ldots$$

How are the $\gamma_{i+1} - \gamma_i$'s distributed?

First, we need to "unfold" the zeros

$$N(T) \sim \frac{T}{2\pi} \log\left(\frac{T}{2\pi e}\right),$$



$$\tilde{\gamma}_i = \gamma_i \cdot \frac{\log \gamma_i}{2\pi}$$

$\tilde{\gamma}_{i+1} - \tilde{\gamma}_i$ is on average 1.

Montgomery looked at the pair correlation of the $\tilde{\gamma}_i$'s, a statistic that measures how much nearby zeros know about each other.

Montgomery conjectured

$$\lim_{M\to\infty} \frac{1}{M} \#\{ i,j \le M \mid a \le \tilde{\gamma}_i - \tilde{\gamma}_j \le b \}$$

$$= \boxed{\int_a^b 1 - \left(\frac{\sin \pi x}{\pi x}\right)^2 dx}, \quad 0 < a < b.$$

$\frac{1}{M}$ is the correct normalization:

for any j, there are just a handful of i's with $\tilde{\gamma}_i - \tilde{\gamma}_j \in [a,b]$.

He was able to prove, assuming RH,

$$\frac{1}{M} \sum_{j<i\le M} f(\tilde{\gamma}_i - \tilde{\gamma}_j) \to \int_0^\infty f(x)\left(1 - \left(\frac{\sin \pi x}{\pi x}\right)^2\right) dx$$

for test functions f with certain restrictions (mainly on the support of $\hat{f}$).

Freeman Dyson pointed out that large unitary matrices have the same pair correlation!

Let

$e^{i\theta_1}, e^{i\theta_2}, \ldots e^{i\theta_N}$ be the eigenvalues of a matrix in $U(N)$.

$0 \le \theta_1 \le \theta_2 \ldots \le \theta_N < 2\pi$

$\tilde{\theta}_i = \theta_i \cdot \frac{N}{2\pi}$ so $\tilde{\theta}_{i+1} - \tilde{\theta}_i$ is 1 on average.

$$\frac{1}{N} \sum_{j<i\le N} \chi_{[a,b]}\left(\tilde{\theta}_i - \tilde{\theta}_j\right), \quad 0<a<b$$

equals, as we average over $U(N)$, and let $N \to \infty$,

$$\int_a^b \left(1 - \left(\frac{\sin \pi x}{\pi x}\right)^2\right) dx$$

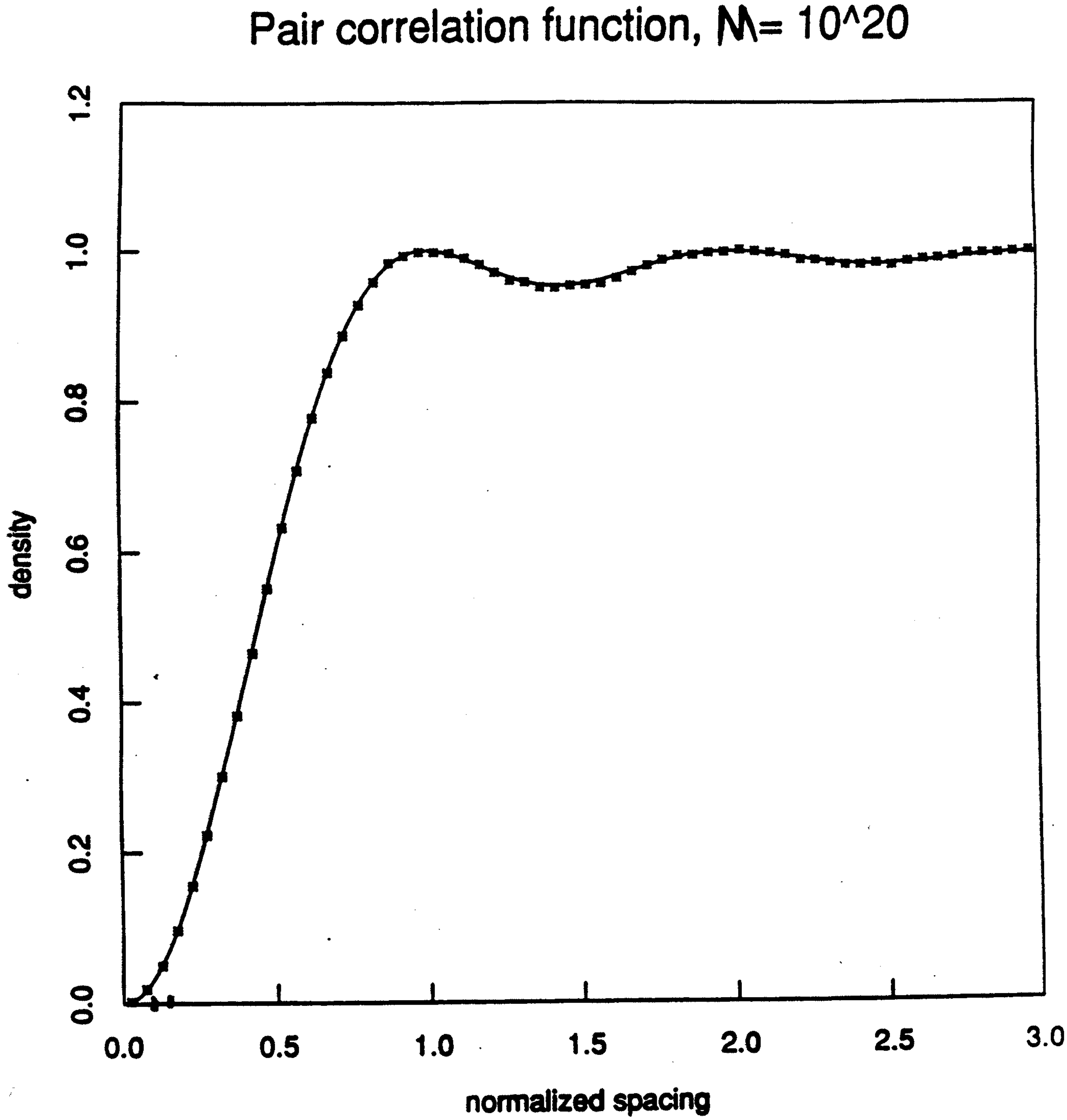


Figure 2.3.1. Pair correlation of zeros of the zeta function. Solid line: GUE prediction. Scatterplot: empirical data based on 8×10^6 zeros near zero number 10^{20}.

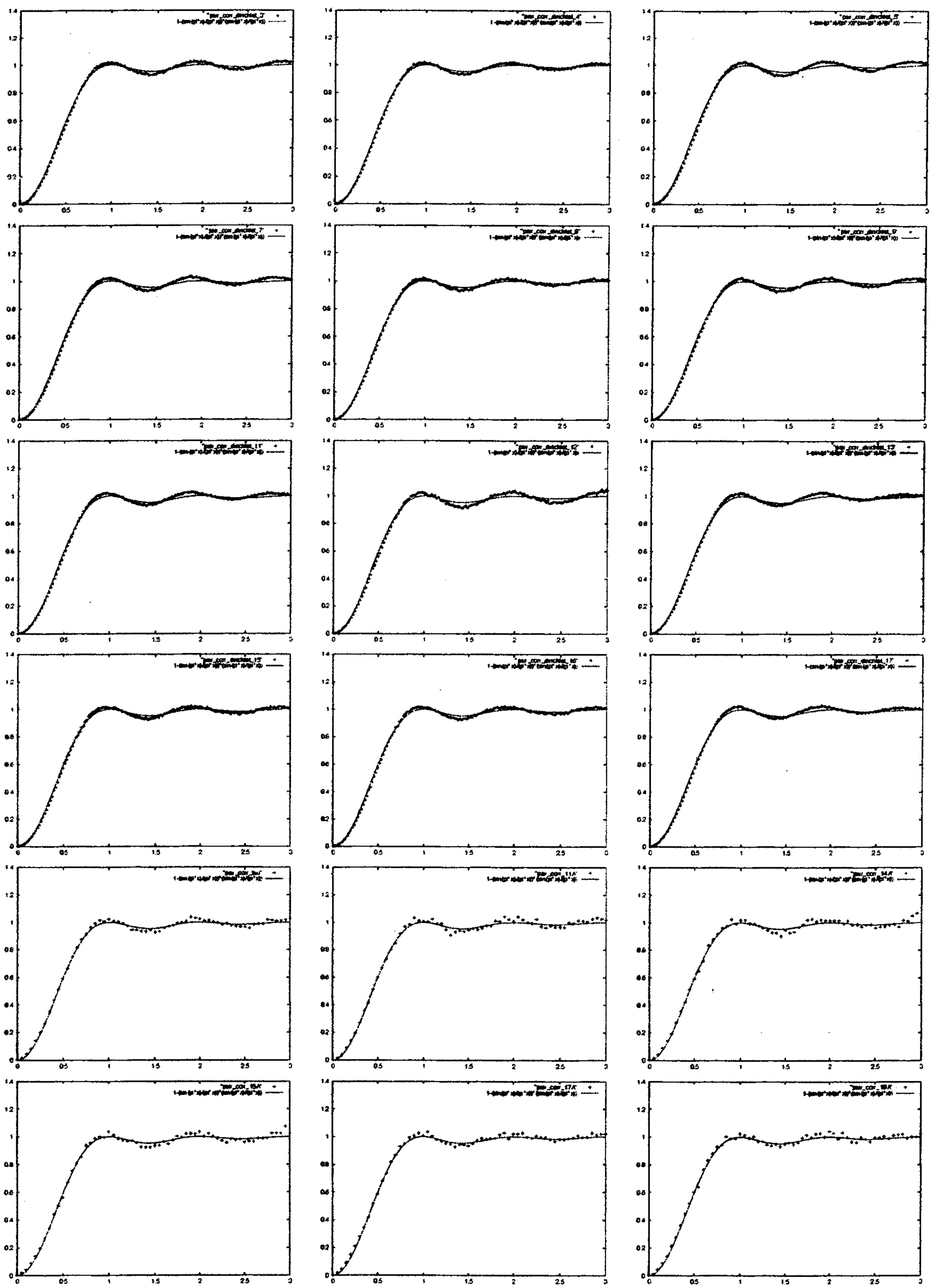


FIGURE 2. Pair correlation for zeros of all primitive $L(s,\chi)$, $3 \le q \le 18$, the Ramanujan τ L-function, and five elliptic curve L-functions

Montgomery's Thm (Rudnick-Sarnak approach)

To apply explicit formula, rather than have a sharp cutoff

$$\frac{1}{M} \sum_{j \neq k \leq M} f(\tilde{\gamma}_j - \tilde{\gamma}_k)$$

we smooth, and also normalize the zeros by the same factor $\frac{\log T}{2\pi}$.

Let $\hat{h}_i$ be smooth, compactly supported, even, so that

$$h_i(y) = \int_{-\infty}^{\infty} \hat{h}_i(t) e^{2\pi i y t} \, dt$$

is smooth, rapidly decreasing, entire

Assume same for f, but with $\hat{f}$ supported in $(-1,1)$

Let

$$R_2(T,f,h) = \sum_{j\neq k} h_1\left(\frac{\gamma_j}{T}\right) h_2\left(\frac{\gamma_k}{T}\right) f\left((\gamma_j-\gamma_k)\frac{\log T}{2\pi}\right)$$

Rudnick - Sarnak (also do higher correlations and for any primitive L-function)

$$R_2(T,f,h) \sim \frac{T\log T}{2\pi}\left(\int_{-\infty}^{\infty} h_1(r)h_2(r)\,dr\right)\underbrace{\int_{-\infty}^{\infty}\left(1-\left(\frac{\sin \pi x}{\pi x}\right)^2\right)f(x)\,dx}_{\text{coincides with pair correlation for eigenvalues of large unitary matrices}}$$

To get rid of $h_1(r)h_2(r)$, approximate $\chi_{[-1,1]^2}$ analytically by a linear comb of such guys.

If we <u>assume RH</u>, then $h_1\left(\frac{\gamma_j}{T}\right)h_2\left(\frac{\gamma_k}{T}\right)$ is evaluated at real values where it approximates

$\chi_{[-1,1]^2}$

Outline of proof

$$f\left((\gamma_i - \gamma_j)\frac{\log T}{2\pi}\right) = \int_{-\infty}^{\infty} \hat{f}(u)\, e^{2\pi i u (\gamma_j - \gamma_k)\frac{\log T}{2\pi}}\, du$$

So

can make this – since h is even and $\gamma_i - \gamma$'s come in pairs.

$$R_2(T, f, h) = \int_{-\infty}^{\infty} \left(\left(\sum_{\gamma} h_1\left(\frac{\gamma}{T}\right) e^{iu\gamma \log T} \right) \cdot \left(\sum_{\gamma} h_2\left(\frac{\gamma}{T}\right) e^{-iu\gamma \log T} \right) - \sum_{\gamma} h_1\left(\frac{\gamma}{T}\right) h_2\left(\frac{\gamma}{T}\right) \right) \hat{f}(u)\, du$$

Apply explicit formula

$$\sum_{\gamma} h\left(\frac{\gamma}{T}\right) e^{-iu\gamma \log T}$$

$$= h\left(\frac{i}{2T}\right)\left(T^{u/2} + T^{-u/2}\right)$$

$$+ \frac{1}{2\pi}\int_{-\infty}^{\infty} h\left(\frac{t}{T}\right) T^{-iuT}\left(-\log \pi + \frac{1}{2}\frac{\Gamma'}{\Gamma}\left(\frac{1}{4}+\frac{it}{2}\right) + \frac{1}{2}\frac{\Gamma'}{\Gamma}\left(\frac{1}{4}-\frac{it}{2}\right)\right) dt$$

$$-\frac{T}{\pi}\sum \frac{\Lambda(n)}{\sqrt{n}}\left(\hat{h}\left(\frac{T}{2\pi}(\log n + u\log T)\right) + \hat{h}\left(\frac{T}{2\pi}(-\log n + u \log T)\right)\right)$$

Plug into previous slide, multiply out.

Complicated mess.

In a nutshell: support condition on $\hat{f}$ restricts us to region where main contribution comes from the diagonal term, $\sum_p \frac{(\log p)^2}{p} h_1(*) h_2(*)$