Number Theory Background, II

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Riemann Zeta-function

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \qquad s = \sigma + it, \qquad \sigma > 1.$$

<u>Motivating</u> <u>Question</u>: How big is $\zeta(s)$?

(answer next Thursday)

$$\zeta(s) \sim \frac{1}{s-1} \quad \text{as } s \to 1^+$$

For $\sigma > 1$, $\qquad \zeta(\sigma + it) = O_\sigma(1)$

- $\zeta(s)$ continues to a meromorphic function
- One simple pole at $s = 1$
- functional equation relates $\zeta(s)$ to $\zeta(1-s)$

Equivalent versions of the functional equation

$$\zeta(s) = \chi(s)\,\zeta(1-s) \qquad \chi(s) = \frac{\pi^{s-\frac{1}{2}}\,\Gamma\left(\frac{1}{2}-\frac{s}{2}\right)}{\Gamma\left(\frac{s}{2}\right)}$$

$$\chi(s) = \frac{1}{\chi(1-s)}$$

$$\left|\chi\left(\tfrac{1}{2}+it\right)\right| = 1$$

$$\xi(s) := \tfrac{1}{2}s(s-1)\,\pi^{-\frac{s}{2}}\,\Gamma\left(\tfrac{s}{2}\right)\zeta(s)$$
$$= \xi(1-s)$$

$$\Xi(z) := \xi\left(\tfrac{1}{2}+iz\right)$$
$$= \Xi(-z)$$

$$Z(t) = \chi\left(\tfrac{1}{2}+it\right)^{-1/2}\zeta\left(\tfrac{1}{2}+it\right) \quad \text{is real for real } t$$

$$\sim C\,\frac{e^{\frac{\pi t}{4}}}{t^2}\cdot\Xi(t) \quad \text{as } t\to\infty$$

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In[11]:= Plot[Abs[Zeta[1/2 + I t]], {t, 1000, 1010}, AspectRatio → 1/3, PlotRange → All];
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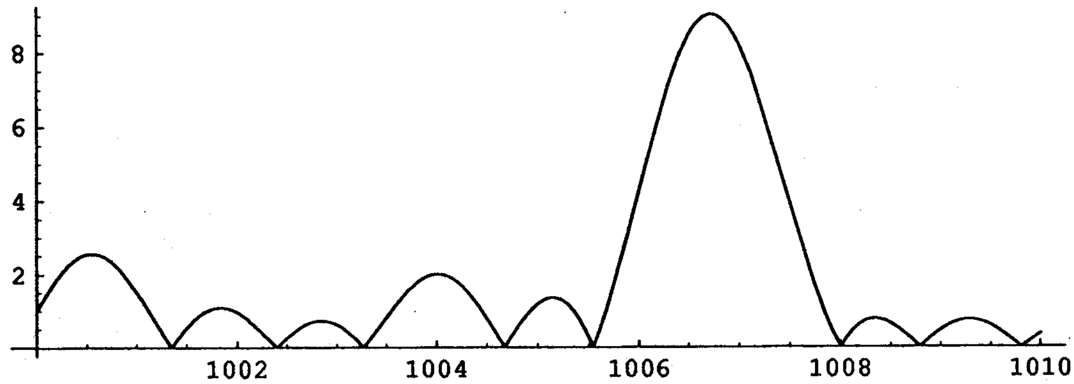


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In[13]:= Plot[Abs[Zeta[1/2 + I t]], {t, 10000, 10010}, AspectRatio → 1/3, PlotRange → All];
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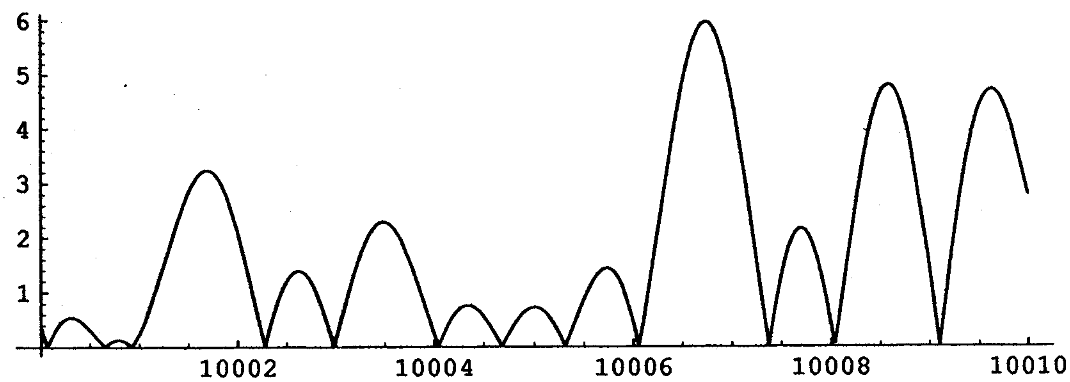


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In[14]:= Plot[Abs[Zeta[1/2 + I t]], {t, 100000, 100010}, AspectRatio → 1/3, PlotRange → All];
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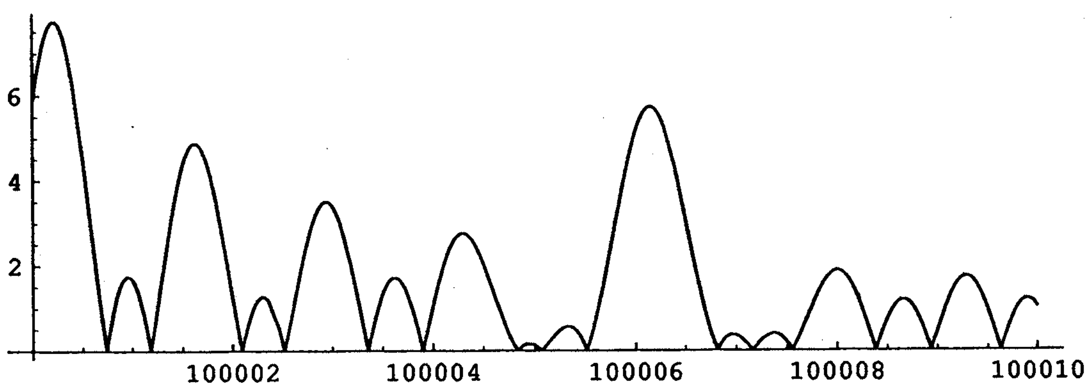

$$N(T) := \#\{\rho = \beta + i\gamma : 0 < \gamma < T\}$$

$$= \frac{1}{2\pi} T \log\left(\frac{T}{2\pi e}\right) + O(\log T)$$

Average gap is $\dfrac{2\pi}{\log T}$

R_2

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In[19]:= ParametricPlot[{Re[Zeta[1/2 + I t]], Im[Zeta[1/2 + I t]]},
           {t, 1000, 1010}, AspectRatio → 1, PlotRange → All];
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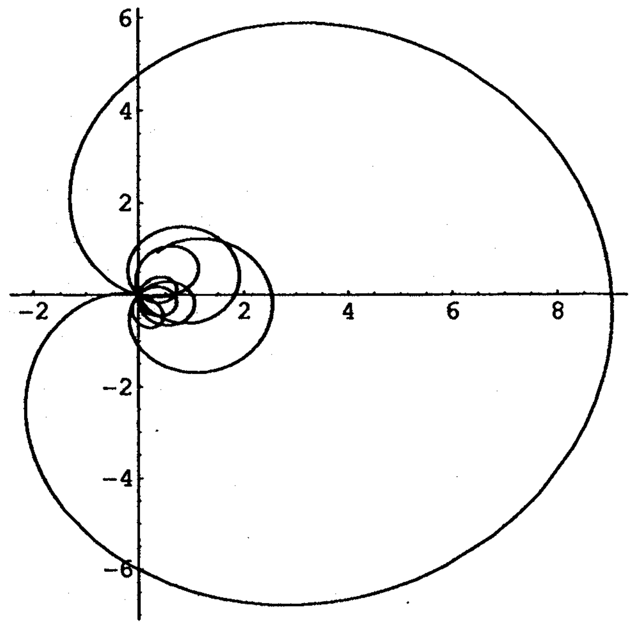


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In[21]:= Plot[RiemannSiegelZ[t], {t, 1000, 1010}, AspectRatio → 1/3, PlotRange → All];
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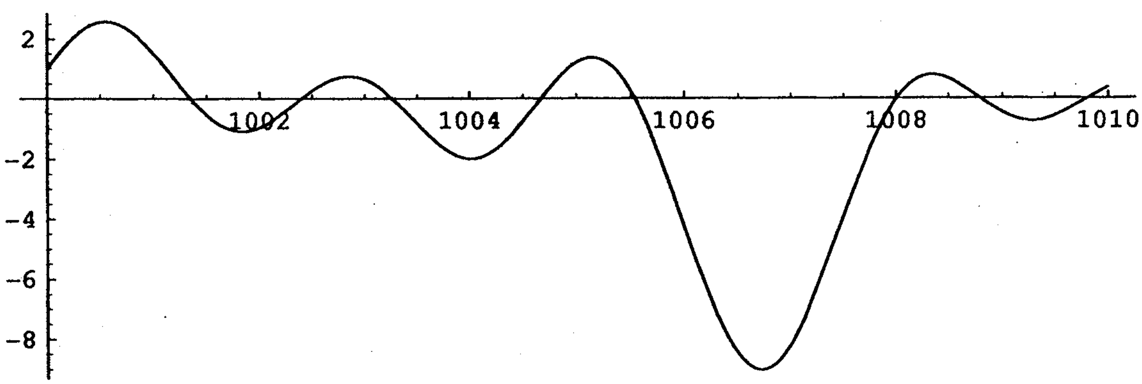

R. (9)

Size of $\zeta(\sigma+it)$ as $t \to \infty$

$\sigma > 1$, $\quad \zeta(\sigma+it) \asymp 1$

For $\sigma < 0$, use $\zeta(s) = \chi(s)\,\zeta(1-s)$

Stirling's formula:

$$|\Gamma(\sigma+it)| = e^{-\frac{\pi t}{2}}\, t^{\sigma-\frac{1}{2}}\left(1+O_\sigma\left(\tfrac{1}{t}\right)\right)$$

$$\chi(s) = \pi^{s-\frac{1}{2}}\,\frac{\Gamma\left(\frac{1}{2}-\frac{s}{2}\right)}{\Gamma\left(\frac{s}{2}\right)}$$

$$|\chi(\sigma+it)| \sim \pi^{\sigma-\frac{1}{2}}\,\frac{\left(\frac{t}{2}\right)^{\frac{1}{2}-\frac{\sigma}{2}-\frac{1}{2}}}{\left(\frac{t}{2}\right)^{\frac{\sigma}{2}-\frac{1}{2}}}$$

$$= \left(\frac{t}{2\pi}\right)^{\frac{1}{2}-\sigma}$$

So $\zeta(\sigma+it) \asymp t^{\frac{1}{2}-\sigma}$ $\quad$ for $\sigma < 0$

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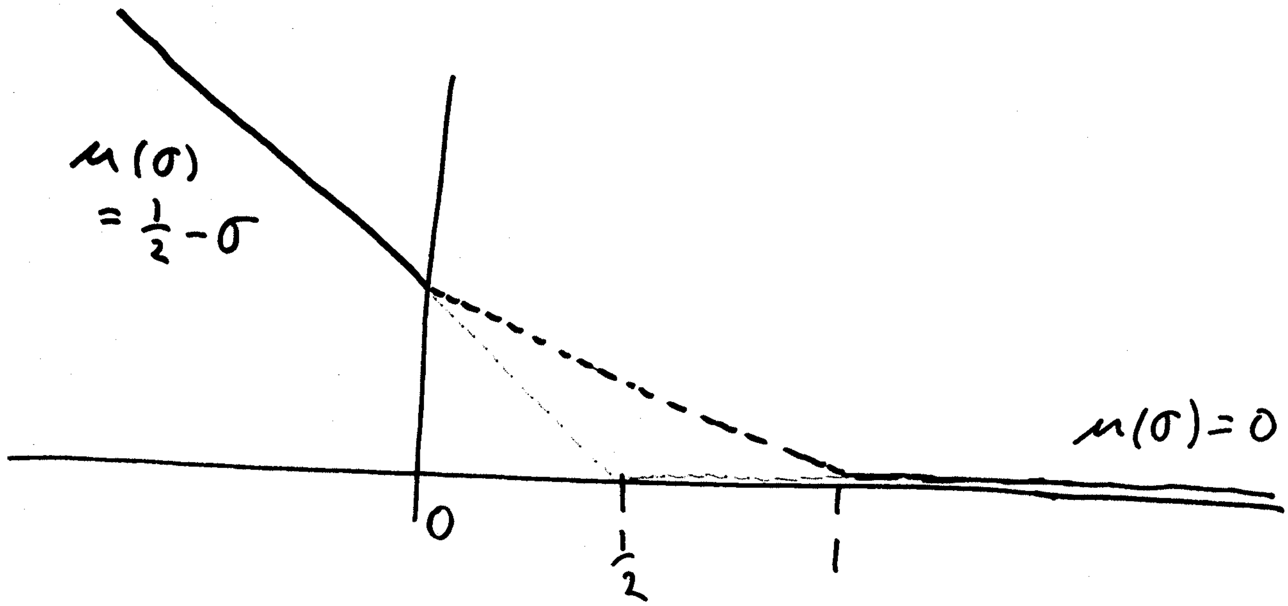
$$\mu(\sigma) = \inf_{\mu} \left|\zeta(\sigma+it)\right| \ll t^{\mu}$$

Phragmen-Lindelöf: μ is a convex functio



$$\zeta(\tfrac{1}{2}+it) \ll t^{\frac{1}{4}+\varepsilon}$$ "Convexity bound" "trivial bound"

$$\zeta(\tfrac{1}{2}+it) \ll t^{\varepsilon}$$ Lindelöf Hypothesis

RH $\Longrightarrow$ Lindelöf

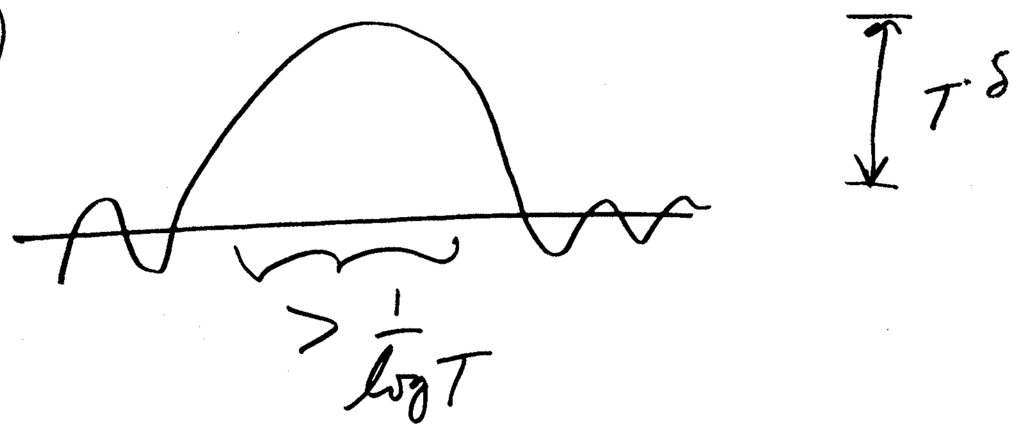
Lindelöf and Moments

Theorem: LH $\iff$ $\displaystyle\int_0^T |\zeta(\tfrac{1}{2}+it)|^{2k}\,dt \ll T^{1+\varepsilon}$

for all $k>0$, $\varepsilon>0$.

Proof: ($\Longrightarrow$) LH $\Rightarrow$ $\zeta(\tfrac{1}{2}+it) \ll t^{\varepsilon/2k}$

($\Longleftarrow$)



So $\displaystyle\int_0^T |\zeta(\tfrac{1}{2}+it)|^{2k}\,dt > \frac{1}{\log T}\, T^{2k\delta}$,

Choose $2k > \dfrac{1+\varepsilon}{\delta}$.

(Exercise: use Cauchy's Theorem and the functional equation to bound $\zeta'(\tfrac{1}{2}+it)$)

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The Bad News: $\int_0^T |\zeta(\tfrac{1}{2}+it)|^{2k}\,dt \ll T^{1+\varepsilon}$

only known for $k = 0$ trivial

1 Hardy-Littlewood

2 Ingham.

New motivation:

Understand (conjecturally) the precise structure of critical moments. Example:

- $\int_0^T |\zeta(\tfrac{1}{2}+it)|^{2k}\,dt$ or $\int_0^T |L(\tfrac{1}{2}+it,\chi)|^{2k}\,dt$

- $\sum_{\chi \bmod q} |L(\tfrac{1}{2},\chi)|^{2k}$

- $\sum_{q \le Q} \sum_{\substack{\chi \bmod q \\ \chi \text{ real} \\ \text{primitive}}} L(\tfrac{1}{2},\chi_q)^k$

Primitive Real Characters

- $$\chi_4(n) = \begin{cases} 1 & n \equiv 1 \mod 4 \\ -1 & n \equiv -1 \mod 4 \end{cases}$$

- Conductor 8: There are 2 of them (Exercise)

p prime

- $$\chi_p(n) = \left(\frac{n}{p}\right) = \begin{cases} 1 & n \equiv \square \mod p \\ -1 & n \not\equiv \square \mod p \\ 0 & n \equiv 0 \mod p \end{cases}$$

Legendre Symbol

<u>Theorem:</u> There exists a real primitive character mod d if and only if

$$d = 2^a p_1 p_2 \cdots p_J, \qquad a = 0, 2, 3, \quad p_j \text{ distinct primes.}$$

(then d is called a "fundamental discriminant")

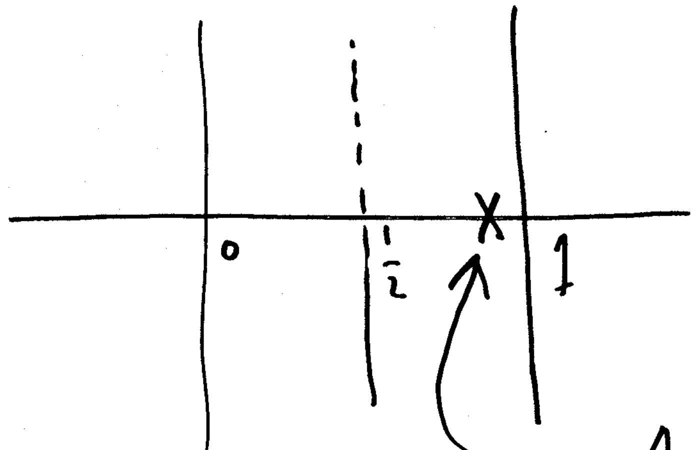
The real primitive characters are of the form

$$\chi_{2^a} \chi_{p_1} \chi_{p_2} \cdots \chi_{p_J}$$

(So there are 1 or 2 of them)

How small can $L(1, \chi_d)$ be?

Can $L(s, \chi_d)$ have a zero very close to $s=1$?



A Landau-Siegel zero

Deuring-Heilbronn, Montgomery, Heath-Brown, Conrey-Iwaniec

If there are Landau-Siegel zeros, then the zeros of other L-functions behave unbelievably strangely.

Conrey & Iwaniec: Spacing of zeros of Hecke L-functions and the class number problem, Acta Arith 103 (2002) 259-312 (paper is on the ArXiv).

Assume RH, zeros of zeta: $\frac{1}{2} + i\gamma$

$0 < \gamma_1 < \gamma_2 < \cdots$

Let $\tilde{\gamma} = \frac{1}{2\pi} \gamma \log \gamma$,

so $\tilde{\gamma}_{j+1} - \tilde{\gamma}_j$ is 1 on average.

Montgomery's motivation for studying pair correlation:

If there are Landau-Siegel zeros, then for many zeros of $\zeta(s)$, the normalized gap $\tilde{\gamma}_{j+1} - \tilde{\gamma}_j$ must be very close to an positive integer or half-integer

(see Conrey-Iwaniec for a precise statement)

Sufficient to show $\tilde{\gamma}_{j+1} - \tilde{\gamma}_j < 0.49999$ for a positive proportion of j

Montgomery $\tilde{\gamma}_{j+1} - \tilde{\gamma}_j < 0.68$ " " " "

Current record: $< 0.51\ldots$

New motivation:

Understand (conjecturally) the statistical properties of zeros of L-functions.

- The sequence of zeros $\gamma_1 \le \gamma_2 \le \gamma_3 \le \cdots$ of a single L-function
- The zeros near the critical point of $L(s, \chi_d)$

Random Matrix Theory will shed light on

- The moments of L-functions
- The statistics of ~~zeros~~ of L-functions.

Dirichlet's class number formula

$K = \mathbb{Q}(\sqrt{d})$ $\qquad\qquad d \equiv 0, 1 \mod 4$

χ_d real primitive character associated to K

$$\zeta_K(s) = \text{Hecke } L\text{-function of } K$$

$$= \zeta(s)\, L(s, \chi_d)$$

$$L(1, \chi_d) = \begin{cases} \dfrac{2\pi h}{\omega \sqrt{-d}} & d < 0 \\[2ex] \dfrac{2h \log \varepsilon}{\sqrt{d}} & d > 0 \end{cases}$$

Where h = class number of $\mathbb{Q}(\sqrt{d})$

= a positive integer

ω = # of units of K

ε = fundamental unit of K

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