Number Theory Background, I

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References:

Analytic Number Theory (AMS)
by Iwaniec & Kowalski

Intro. to Analytic Number Theory (Springer)
by Apostol

Multiplicative Number Theory (Springer)
by Davenport

The zeta-function (according to Euler 1737)

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

$$= \prod_p \left(1 - \frac{1}{p^s}\right)^{-1} \qquad \text{Euler product}$$

$\prod_p$ is a product over all primes

Sum and product converge for $s > 1$.

<u>Theorem</u> $\sum_p \frac{1}{p}$ diverges

<u>Proof</u>: $\log \zeta(s) = -\sum_p \log\left(1 - \frac{1}{p^s}\right)$

$$= \sum_p \frac{1}{p^s} + O\Big(\underbrace{\sum \frac{1}{p^{2s}}}_{\text{bounded}}\Big)$$

diverges

as $s \to 1^+$

L-functions (Dirichlet 1832)

$$L(s) = 1 - \frac{1}{3^s} + \frac{1}{5^s} - \frac{1}{7^s} + \frac{1}{9^s} - + - \cdots$$

$$= \sum_{n=1}^{\infty} \frac{\chi_4(n)}{n^s} \qquad \chi_4(n) = \begin{cases} -1 & n \equiv 3\ (4) \\ 1 & n \equiv 1\ (4) \\ 0 & \text{o.w.} \end{cases}$$

$$= \prod_{p \equiv 1 \pmod 4} \left(1 - \frac{1}{p^s}\right)^{-1} \prod_{p \equiv 3 \pmod 4} \left(1 + \frac{1}{p^s}\right)^{-1}$$

$$\log L(s) = \sum_{p \equiv 1 (4)} \frac{1}{p^s} - \sum_{p \equiv 3 (4)} \frac{1}{p^s} + \text{l.o.t.}$$

Dirichlet: $L(1)$ converges. If $L(1) \neq 0$ then $\log L(s)$ is bounded as $s \to 1^+$, so both of

$$\log \zeta(s) + \log L(s) = \sum_{p \equiv 1 (4)} \frac{1}{p^s}$$

$$\log \zeta(s) - \log L(s) = \sum_{p \equiv 3 (4)} \frac{1}{p^s}$$

diverge as $s \to 1^+$.

So $\sum \frac{1}{p}$ and $\sum \frac{1}{p}$ diverge

R, ②

χ is a <u>Dirichlet character</u> modulo q if

$$\chi : \mathbb{Z} \to \mathbb{C}$$

$\chi(nm) = \chi(n)\chi(m)$ for all n, m

"χ is completely multiplicative"

$\chi(n+q) = \chi(n)$

$\chi(m) = 0$ if $(m,q) > 1$.

ie, $\chi : (\mathbb{Z}/q\mathbb{Z})^* \to \mathbb{C}$

extend to $\mathbb{Z}$ by $\chi(m) = 0$ if $(m,q) \neq 1$.

5	1	2	3	4
χ_0	1	1	1	1
χ_5	1	-1	-1	1
χ	1	i	$-i$	-1
$\bar{\chi}$	1	$-i$	i	-1

Characters mod 5

6	1	5
χ_0	1	1
χ	1	-1

Characters mod 6

$\varphi(q)$ Characters

χ_0 trivial character

Orthogonality:

$$\frac{1}{\varphi(q)} \sum_{n(q)} \chi(n) = \begin{cases} 1 & \chi = \chi_0 \\ 0 & \text{o.w.} \end{cases}$$

$$\frac{1}{\varphi(q)} \sum_{\chi(q)} \chi(n) = \begin{cases} 1 & n \equiv 1\ (q) \\ 0 & \text{o.w.} \end{cases}$$

Primitivity:

Some real characters mod 20

20	1	3	7	9	11	13	17	19	
$\chi_4 \chi_0$	1	-1	-1	1	-1	1	1	-1	not primitive
$\chi_5 \chi_0$	1	-1	-1	1	1	-1	-1	1	
χ_{20}	1	1	1	1	-1	-1	-1	-1	

" χ_5 induces a character mod 20 "

primitive $\Longleftrightarrow$ not induced from a smaller modulus

"conductor" = smallest modulus induced from

primitive $\Longleftrightarrow$ conductor = modulus

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Dirichlet, continued

$$L(s,\chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s} = \prod_p \left(1 - \frac{\chi(p)}{p^s}\right)^{-1}$$

By orthogonality, $\exists\, c_\chi$ s.t.

$$\log \zeta(s) + \sum_\chi c_\chi \log L(s,\chi) = \sum_{p \equiv a(q)} \frac{1}{p^s} + \text{l.o.t.}$$

So $L(1,\chi) \neq 0 \Rightarrow \sum_{p \equiv a(q)} \frac{1}{p}$ diverges.

<u>Exercise</u>: What is c_χ?

Hardest part of Dirichlet's proof:

$L(1,\chi) \neq 0$ for <u>real</u> χ

"quadratic character"

$\chi^2 = \chi_0$

The zeta-function (according to Riemann 1859)

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

$$= \prod_p \left(1 - \frac{1}{p^s}\right)^{-1}$$

$s = \sigma + it$ is a complex variable

Sum and product converge for $\sigma > 1$

<u>Theorem</u> ("Riemann")

$$\pi(x) := \#\{p \leq x : p \text{ prime}\}$$

$$\sim \frac{x}{\log x}$$

$$\Updownarrow$$

$$\zeta(1+it) \neq 0 \quad \text{for any } t \in \mathbb{R}$$

Similarly, $L(1+it, \chi) \neq 0$ implies

$$\pi(x; a, q) := \#\{p \leq x : p \equiv a \bmod q\}$$

$$\sim \frac{1}{\varphi(q)} \frac{x}{\log x}$$ if $(a,q)=$

Analytic Continuation & Functional Equation

χ primitive character mod q (allow $q=1$)

$$K = \begin{cases} 0 & \chi \text{ even} \\ 1 & \chi \text{ odd} \end{cases}$$

$$L(s,\chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s} = \prod_p \left(1 - \frac{\chi(p)}{p^s}\right)^{-1}$$

Continues to a meromorphic function (entire if $q \neq 1$) which satisfies the functional equation

$$\Lambda(s,\chi) := \left(\frac{q}{\pi}\right)^{s/2} \Gamma\left(\frac{s+K}{2}\right) L(s,\chi)$$

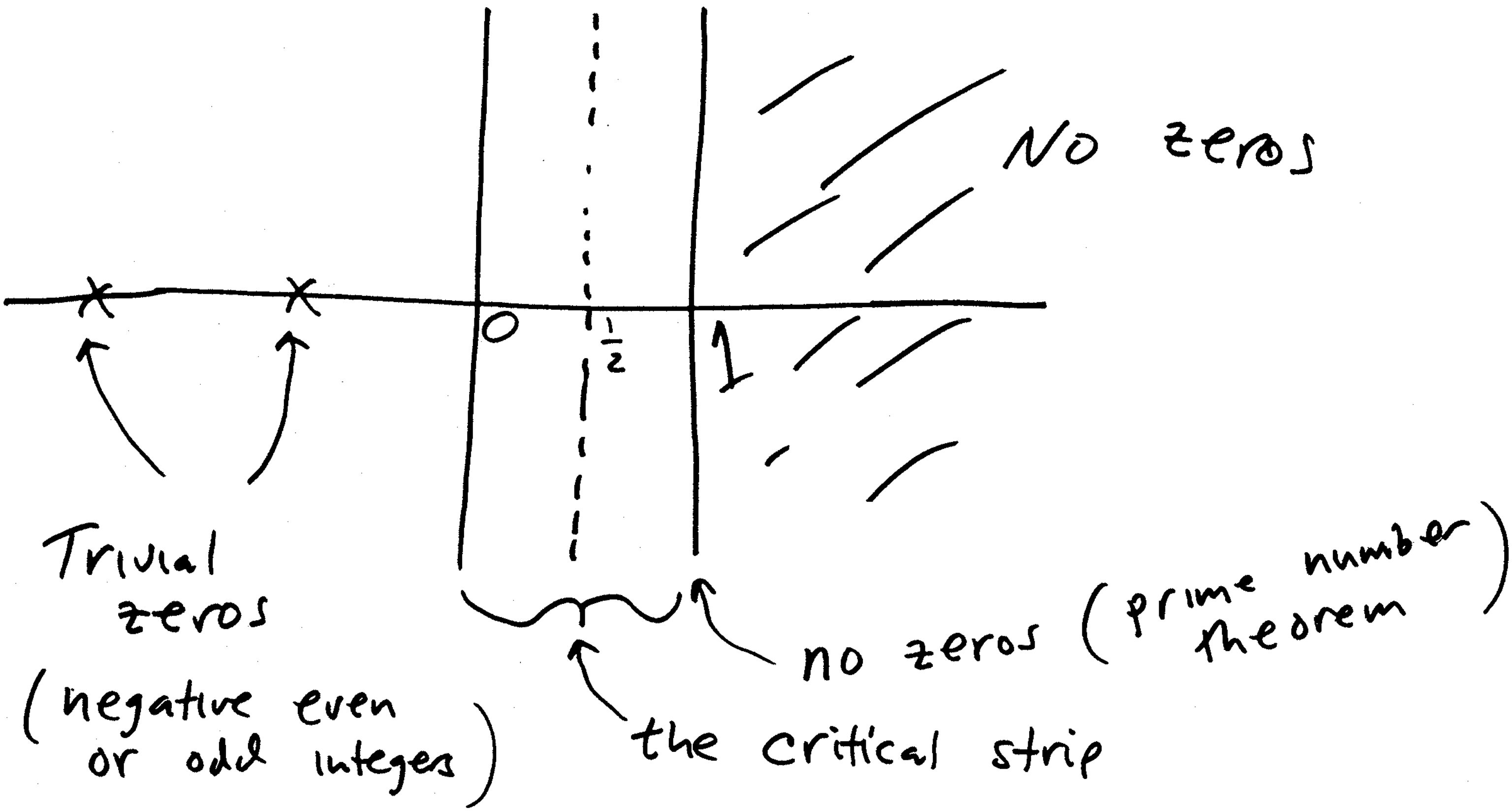
$$= \varepsilon(\chi)\, \Lambda(1-s, \bar{\chi})$$

Where $\varepsilon(\chi) = i^{-K} \dfrac{\tau(\chi)}{\sqrt{q}}$

with $\tau(\chi) = \sum_{b \bmod q} \chi(b)\, e^{\frac{2\pi i b}{q}}$ Gauss sum.

$q \neq 1$, $\Lambda(s,\chi)$ entire
$\Lambda(s,1)$ poles at $s = 0, 1$

$\Gamma(s)$ no zeros; poles at $0, -1, -2, \ldots$



$$Li(x) := \int_2^x \frac{dt}{\log t} \quad \text{the Logarithmic integral}$$

$$\sim \frac{x}{\log x} + \frac{x}{\log^2 x} + \frac{2!\,x}{\log^3 x} + \cdots + \frac{m!\,x}{\log^{m+1} x} + \cdots$$

(asymptotic series)

PNT: $\pi(x) \sim Li(x)$

RH: $\pi(x) = Li(x) + O(x^{\frac{1}{2}+\varepsilon})$

(Details in Rubinstein's lecture)

RH: Nontrivial zeros on the critical line $Re(s) = \frac{1}{2}$

Exercises:

1) Write down the functional equation for nonprimitive $L(s, \chi)$

(conductor $= q <$ m = modulus)

2) Show that if χ nonprimitive then RH is false for $L(s,\chi)$.

The two main questions:

How fast does the zeta-function grow?

$L(1,\chi) > 0$. How small can it be?