# MOMENTS II

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## 1. Mean Values (or Moments)

Moments are averages of the modulus of a function raised to a power. For example,

$$\frac{1}{2\pi}\int_0^{2\pi} |f(z_0 + Re^{i\theta})|^{2k}\, d\theta\,.$$

For functions represented by a Dirichlet series in some half–plane $\Re s > \sigma_0$ of the complex plane, the average is usually over a vertical segment:

$$\int_T^{2T} |F(\sigma+it)|^{2k}\, dt.$$

**Note:**
1) The path of integration need not lie in the half–plane of convergence.
2) It is customary not to divide by $T$.

There are many variants. For example, the average can be over a discrete set of points $\sigma_r + it_r \in \mathbb{C}$:

$$\sum_{r=1}^{R} |F(\sigma_r + it_r)|^{2k}\,.$$

**Examples**
1) We would like to estimate the moments

$$I_k(\sigma, T) = \int_T^{2T} |\zeta(\sigma+it)|^{2k}\, dt\,,$$

for $\sigma \geq \frac{1}{2}$. Note that the Dirichlet series

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$$

converges only when $\sigma > 1$.
2) Let $\rho = \beta + i\gamma$ run over the zeros of the zeta-function and let $\zeta^{(j)}$ denote the $j$th derivative of $\zeta$. Then

$$J_k(T) = \sum_{0<\gamma\leq T} |\zeta^{(j)}(\rho)|^{2k}$$

is an example of a discrete mean.

## 2. Why Moments?

Moment estimates are useful for studying the zeros of the zeta function and other L-functions and we will see such applications in the next lecture. Two more obvious reasons are:
1) If a high moment is "small", the function cannot be very "big" (or always "big").
2) If a high moment is "big", the function cannot always be "small".
Here is an example of 1.
**Example:**
Suppose that

$$I_k(\tfrac{1}{2},T) = \int_T^{2T} \left|\zeta(\tfrac{1}{2}+it)\right|^{2k} dt \ll T^A.$$

It is known that $\zeta'(\frac{1}{2}+it) \ll t^{1/4}$. So if $|\zeta(\frac{1}{2}+it)|$ takes its maximum on $[0,T]$ at $t_0$, then $|\zeta(\frac{1}{2}+it)| \geq \frac{1}{2}|\zeta(\frac{1}{2}+it_0)|$ on an interval of length $\gg t^{-1/4}$. Hence

$$|\zeta(\tfrac{1}{2}+it_0)|^{2k} t^{-1/4} \ll \int_T^{2T} \left|\zeta(\tfrac{1}{2}+it)\right|^{2k} dt \ll T^A,$$

so on$[T,2T]$

$$\zeta(\tfrac{1}{2}+it) \ll T^{(A+1/4)/2k}.$$

In particular, if $A = 1+\epsilon$ holds for every positive integer $k$, then the Lindelöf Hypothesis, $\zeta(\frac{1}{2}+it) \ll t^{\epsilon}$, holds.
**Example of 2:**
Consider

$$I_1(\sigma,T) = \int_T^{2T} \left|\zeta(\sigma+it)\right|^2 dt\,,$$

We have seen that

$$I_1(\tfrac{1}{2},T) \sim T\log T\,.$$

It is even easier to show that for a fixed $\sigma > \frac{1}{2}$

$$I_1(\sigma,T) \sim \zeta(2\sigma)T.$$

Thus $|\zeta(s)|$ is larger on the one-half line than to the right of it. Since it also has zeros on the one-half line, zeta should be more erratic there then to the right.

## 3. What we know

Recall that

$$I_k(\tfrac{1}{2},T) = \int_0^{T} \left|\zeta(\tfrac{1}{2}+it)\right|^{2k} dt\,.$$

As we have seen, in 1918 Hardy and Littlewood proved that

$$I_1(\tfrac{1}{2},T) \sim T\log T\,.$$

The next higher moment was detrermined in 1926 by Ingham, ,who proved that

$$I_2(\tfrac{1}{2},T) \sim \frac{T}{2\pi^2}\log^4 T\,.$$

No asymptotic estimate for any other $k \neq 0$ has ever been proved.

Ramachandra and Heath–Brown have shown that

$$I_k(\tfrac{1}{2},T) \gg T\log^{k^2}T$$

for all rational $k>0$. Ramachandra showed it for all $k>0$ if the Riemann Hypothesis is true.

Heath–Brown showed that for $k=\frac{1}{n}$

$$T\log^{k^2}T \ll I_k(\tfrac{1}{2},T) \ll T\log^{k^2}T\,.$$

Ramachandra did the case $k=\frac{1}{2}$ first.

Heath–Brown also showed that on the Riemann Hypothesis

$$I_k(\tfrac{1}{2},T) \ll T\log^{k^2}T \qquad (0\le k\le 2)\,.$$

Thus on RH

$$T\log^{k^2}T \ll I_k(\tfrac{1}{2},T) \ll T\log^{k^2}T$$

for all $0\le k\le 2$.

We expect that

$$I_k(\tfrac{1}{2},T) = c_kT\log^{k^2}T\,,$$

but a proof seems a long way off, and people worked for many years to find a plausible guess for the value of $c_k$.

Conrey and Ghosh gave a nice proof of the lower bound for all $k\ge 0$ under the assumption of RH, which I will present now. Consider

$$\begin{aligned}
0 \le &\int_T^{2T}\left|\zeta(\tfrac{1}{2}+it)^k - \sum_{n=1}^{T} d_k(n)n^{-(\frac{1}{2}+it)}\right|^2 dt\\
=&I_k(\tfrac{1}{2},T) - 2Re\int_T^{2T}\zeta(\tfrac{1}{2}+it)^k\sum_{n=1}^{T}\frac{d_k(n)}{n^{\frac{1}{2}-it)}}\\
+&\int_T^{2T}\left|\sum_{n=1}^{T}\frac{d_k(n)}{n^{\frac{1}{2}+it)}}\right|^2 dt\,.
\end{aligned}\tag{3.1}$$

By the mean value theorem for Dirichlet polynomials the third term is

$$=\sum_{n=1}^{T}\frac{d_k^2(n)}{n}\,(T+O(n)) = \frac{a_k}{\Gamma(k^2+1)}T\log^{k^2}T + O\left(T\log^{k^2-1}T\right)\,,$$

where

$$a_k = \prod_p\left(\left(1-\frac{1}{p}\right)^{(k-1)^2}\sum_{r=0}^{k-1}\binom{k-1}{r}^2p^{-r}\right)\,.$$

To treat the second term we write

$$\int_T^{2T}\zeta(\tfrac{1}{2}+it)^k\sum_{n=1}^{T}\frac{d_k(n)}{n^{\frac{1}{2}-it}}\,dt = \frac{1}{i}\int_T^{2T}\zeta(s)^k\sum_{n=1}^{T}\frac{d_k(n)}{n^{1-s}}\,ds\,.$$

where $\Re s = \frac{1}{2}$. Next we pull the line of integration right to $Res = c > 1$. There are top and bottom edges of the rectangle to estimate, but these are small. We then have

$$\begin{aligned}\frac{1}{i}\int_T^{2T} \zeta(s)^k \left(\sum_{n=1}^{T} \frac{d_k(n)}{n^{1-s}}\right) ds &= \sum_{n=1}^{T} \frac{d_k(n)}{n} \left(\frac{1}{i}\int_T^{2T} \zeta(s)^k n^s ds\right) \\ &= \sum_{n=1}^{T} \frac{d_k(n)}{n} \sum_{n=1}^{\infty} d_k(m) \left(\frac{1}{i}\int_T^{2T} \left(\frac{n}{m}\right)^s ds\right) \\ &\sim T \sum_{n=1}^{T} \frac{d_k^2(n)}{n} \sim \frac{a_k}{\Gamma(k^2+1)} T \log^{k^2} T\,.\end{aligned}$$

Using this and our previous estimate in (3.1), we find that

$$I_k(\tfrac{1}{2}, T) \geq (1+o(1)) \frac{a_k}{\Gamma(k^2+1)} T \log^{k^2} T\,.$$

If we compare this with Heath–Brown's result (also on RH) with explicit constants, namely

$$I_k(\tfrac{1}{2}, T) \leq (1+o(1)) \frac{a_k}{\Gamma(k^2+1)} \left(\frac{2}{(1+k^2)(2-k)}\right) T \log^{k^2} T$$

for $0 \leq k < 2$, we note that the arithmetical constant $a_k/\Gamma(k^2+1)$ occurs in both. This may have led Conrey and Ghosh to conjecture that the constant $c_k$ in the conjectural formula

$$I_k(\tfrac{1}{2}, T) \sim c_k T \log^{k^2} T$$

factors as

$$c_k = \frac{a_k g_k}{\Gamma(k^2+1)}$$

for some constant $g_k$. But what $g_k$ should be was a mystery. (Note: every number can be factored this way; the point of course is that the $\gamma$ and $a_k$ occur naturally.)

I will conclude this section by mentioning just a few more results.

Gonek showed that on RH

$$I_k(\tfrac{1}{2}, T) \gg T \log^{k^2} T$$

for $-\frac{1}{2} < k < 0$ and that near the one-half line

$$I_k(\sigma, T) \gg T \min(1, (\sigma - \tfrac{1}{2})^{-k^2})$$

for all $k < 0$.

There are also precise results known or conjectured for various naturally occurring discrete moments. For example Gonek gave the formulas

$$\sum_{0<\gamma<T} \zeta^{(\mu)}(\rho)\zeta^{(\nu)}(1-\rho) \sim C_{\mu,\nu} T \log^{\mu+\nu+2} T$$

and

$$\sum_{0<\gamma<T} \zeta(\rho + i\alpha)\zeta(1-\rho-i\alpha) \sim B(\alpha) T \log^2 T\,.$$

On the Riemann Hypothesis one obtains asymptotic formulas for

$$\sum_{0<\gamma<T} |\zeta^{(\mu)}(\tfrac{1}{2} + i\gamma)|^2\,,$$

and

$$\sum_{0<\gamma<T} |\zeta(\tfrac{1}{2}+i\gamma+i\alpha)|^2 \,.$$

Recently Nathan Ng has found close upper and lower bounds on RH for

$$\sum_{0<\gamma<T} |\zeta'(\tfrac{1}{2}+i\gamma)|^4 \,.$$

The question of negative moments arises in this context too.

Gonek showed that on RH

$$\sum_{0<\gamma<T} |\zeta'(\tfrac{1}{2}+i\gamma)|^{-2} \gg T \,,$$

and conjectured that

$$\sum_{0<\gamma<T} |\zeta'(\tfrac{1}{2}+i\gamma)|^{-2} \sim \frac{3}{\pi^3} T \,.$$

Finally we mention that there are results for integrals of the form

$$\int_0^T \left|\zeta^{(j)}(\sigma+it) M_N(\sigma+it)\right|^2 dt$$

and sums of the form

$$\sum_{0<\gamma<T} \left|\zeta^{'}(\rho) M_N(\rho)\right|^2$$

under various hypotheses. Here $M_N$ is a Dirichlet polynomial. For example,

$$M_N(s) = \sum_{1\le n\le N} \frac{\mu(n)}{n^s} P(\frac{\log n}{\log N})$$

with $P(x)$ a polynomial has numerous applications. Balasubramanian, Conrey, Ghosh, Gonek, and Heath–Brown are some of the people who have worked in this area.

## 4. Moments of Dirichlet Polynomials

We have seen that essential ingredients in the proof of the second and fourth moments $I_1(\frac{1}{2}, T)$ and $I_2(\frac{1}{2}, T)$ were approximations to $\zeta(\frac{1}{2}+it)$ by Dirichlet polynomials (approximate functional equations) and the calculation of mean values of those Dirichlet polynomials. Why are we stuck at the fourth moment?

Let

$$A(s) = A_N(s) = \sum_{n=1}^{N} a_n n^{-s}$$

be a Dirichlet polynomial of "length" $N$ with $s = \sigma + it$ and recall the Mean Value Therorem of Montgomery and Vaughan

**Theorem 4.1. (Montgomery–Vaughan)**

$$\int_0^T |\sum_{n=1}^{N} a_n n^{-s}|^2 dt = \sum_{n=1}^{N} |a_n|^2 \, n^{-2\sigma} \Big(T + O(n)\Big) \,.$$

From this we see that if $N = o(T)$, then

$$\int_0^T |\sum_{n=1}^N a_n n^{-\sigma - it}|^2 dt \sim T \sum_{n=1}^N |a_n|^2 n^{-2\sigma}$$

On the other hand, if $N \gg T$ the $O$–term dominates and we have only

$$\int_0^T |\sum_{n=1}^N a_n n^{-\sigma - it}|^2 dt \ll N \sum_{n=1}^N |a_n|^2 n^{-2\sigma}$$

As long as the polynomial has length $\ll T$, we can get an asymptotic main term. Recall that the relevant polynomials for the second and fourth moments were

$$\sum_{n \le \sqrt{T/2\pi}} n^{-s} \qquad \text{and} \qquad \sum_{n \le T/2\pi} d_k(n) n^{-s} .$$

For the 6th, 8th, and higher moments the corresponding approximations require lengths $T^{3/2}, T^2, T^{5/2}, \ldots$.

So one can ask whether there is a precise formula for

$$\int_0^T |\sum_{n=1}^N a_n n^{-s}|^2 dt$$

when $N = T^\alpha$ , $\alpha > 1$.

To answer this, let us square out and integrate a polynomial:

$$\begin{aligned}\int_T^{2T} |\sum_{n=1}^N a_n n^{-\sigma-it}\Big|^2 dt =& \sum_{n=1}^N \sum_{m=1}^N \frac{a_n \overline{a}_m}{(nm)^\sigma} \int_0^T (m/n)^{it}\, dt \\ =& T \sum_{n=1}^N \frac{|a_n|^2}{n^{2\sigma}} + \sum_{\substack{1 \le m,n \le N \\ m \ne n}} \frac{a_n \overline{a}_m}{(nm)^\sigma} \left( \frac{e^{i2T \log(m/n)} - e^{iT \log(m/n)}}{i \log(m/n)} \right) .\end{aligned}$$

We see that the main term in the Montgomery-Vaughan mean value theorem comes from the diagonal terms when $N \ll T$. Therefore, the rest is from the "off–diagonal" terms.

Clearly, if we want a more precise formula when $N$ is larger than $T$, we need to estimate the off–diagonal terms carefully. We see that these terms are

$$\begin{aligned} &2Re \sum_{1 \le n < m \le N} \frac{a_n \overline{a}_m}{(nm)^\sigma} \left( \frac{e^{i2T \log(m/n)} - e^{iT \log(m/n)}}{i \log(m/n)} \right) \\ &= 2Re \sum_{1 \le n < N} \sum_{1 \le h \le N-n} \frac{a_n \overline{a}_{n+h}}{(n(n+h))^\sigma} \left( \frac{e^{i2T \log((n+h)/n)} - e^{iT \log((n+h)/n)}}{i \log((n+h)/n)} \right) \\ &= 2Re \sum_{1 \le h < N} \sum_{1 \le n \le N-h} \frac{a_n \overline{a}_{n+h}}{n^{2\sigma}} (1 + h/n)^\sigma \left( \frac{e^{i2T \log(1+h/n)} - e^{iT \log(1+h/n)}}{i \log(1 + h/n)} \right) .\end{aligned}$$

For the sake of simplicity, consider only the terms with $h/n < 1/2$. In these we have $\log(1+h/n) \approx h/n$ and $(1+h/n)^\sigma \approx 1$. These therefore contribute approximately

$$2Re \sum_{1\le h<N} \sum_{2h<n\le N-h} \frac{a_n\overline{a}_{n+h}}{n^{2\sigma-1}} \left( \frac{e^{i2Th/n} - e^{iTh/n}}{ih} \right) .$$

To simplify further, we restrict our attention to the terms with $Th < n/4$. In these we have $(e^{i2Th/n} - e^{iTh/n})/ih \approx T/n$, so their contribution is

$$\approx 2T\, Re \sum_{h\neq 0} \sum_{n} a_n\overline{a}_{n+h} n^{-2\sigma} .$$

We see that we could estimate this if we had good estimates for the sums

$$\sum_{n=1}^{N} a_n\, \overline{a}_{n+h} .$$

This would be sufficient to estimate the terms we ignored as well.

Goldston and Gonek have worked this out and found a general (although complicated) formula for such sums. Rather than state it, we just reiterate that the crucial ingredient is a good formula for the coefficient correlation sums displayed just above.

## 5. The 6th and 8th Power Moments of the Zeta Function

In the mid 1990's J. B. Conrey and A. Ghosh announced the

**Conjecture 5.1. (Conrey–Ghosh)** *As $T \to \infty$,*

$$\int_0^T |\zeta(\tfrac{1}{2}+it)|^6 \, dt \sim \frac{42}{9!} \prod_p \left( \sum_{r=0}^{\infty} \frac{d_3(p^r)^2}{p^r} \right) T \log^9 T ,$$

*where $d_3(n)$ denotes the number of ways to write $n$ as a product of three positive integers.*

A few years later, J. B. Conrey and I made the following

**Conjecture 5.2. (Conrey–Gonek)** *As $T \to \infty$,*

$$\int_0^T |\zeta(\tfrac{1}{2}+it)|^8 \, dt \sim \frac{24024}{16!} \prod_p \left( \sum_{r=0}^{\infty} \frac{d_4(p^r)^2}{p^r} \right) T \log^{16} T ,$$

*where $d_4(n)$ is the four–fold divisor function.*

We sketch our argument for the 8th moment conjecture. The method gives the 2nd, 4th, and (conjectured) 6th moment asymptotics as well. We begin with a discussion of the approximate functional equation.

For $s = \sigma + it$ and $\sigma > 1$, $\zeta^k(s)$ has the Dirichlet series expansion

$$\zeta^k(s) = \prod_p \left(1 - p^{-s}\right)^{-k} = \prod_p \left(1 + \frac{d_k(p)}{p^s} + \frac{d_k(p^2)}{p^{2s}} + \cdots\right) = \sum_{n=1}^{\infty} \frac{d_k(n)}{n^s} ,$$

where $d_k(p^j) = (-1)^j\binom{-k}{j}$ is the $k$th divisor function. The series does not converge when $\sigma \le 1$, but we can approximate $\zeta^k(s)$ in this region by an expression of the form

$$\zeta(s)^k = \sum_{n=1}^{N} \frac{d_k(n)}{n^s} + \chi(s)^k \sum_{n=1}^{M} \frac{d_k(n)}{n^{1-s}} + E_k(s)\,,$$

where $E_k(s)$ denotes an error term. This is an approximate functional equation. We write it as

$$\zeta(s)^k = \mathbf{D}_{k,N}(s) + \chi(s)^k \mathbf{D}_{k,M}(1-s) + E_k(s)\,,$$

where

$$\mathbf{D}_{k,N}(s) = \sum_{n=1}^{N} \frac{d_k(n)}{n^s}\,,$$

$MN = \left(\frac{t}{2\pi}\right)^k$, and $\chi(s) = \pi^{s-\frac{1}{2}}\Gamma(\frac{1-s}{2})/\Gamma(\frac{s}{2})$ is the factor from the functional equation for the zeta function, $\zeta(s) = \chi(s)\zeta(1-s)$. Taking $s = \frac{1}{2} + it$, we find that

$$\zeta(\tfrac{1}{2}+it)^k = \mathbf{D}_{k,N}(\tfrac{1}{2}+it) + \chi(\tfrac{1}{2}+it)^k \mathbf{D}_{k,M}(\tfrac{1}{2}-it) + E_k(\tfrac{1}{2}+it)\,.$$

Assuming the error term is negligible, we obtain

$$\begin{aligned} \int_T^{2T} |\zeta(\tfrac{1}{2}+it)|^{2k}\,dt \sim \int_T^{2T} |\mathbf{D}_{k,N}(\tfrac{1}{2}+it)|^2\,dt + \int_T^{2T} |\mathbf{D}_{k,M}(\tfrac{1}{2}+it)|^2\,dt \\ + 2Re \int_T^{2T} \chi(\tfrac{1}{2}-it)^k \mathbf{D}_{k,N}(\tfrac{1}{2}+it)\mathbf{D}_{k,M}(\tfrac{1}{2}+it)\,dt\,. \end{aligned} \tag{5.1}$$

There is reason to believe that the cross term is smaller than the main term and that $MN = (t/2\pi)^k$ may be replaced by $MN = (T/2\pi)^k$. Thus, we expect that

$$\int_T^{2T} |\zeta(\tfrac{1}{2}+it)|^{2k}\,dt \sim \int_T^{2T} |\mathbf{D}_{k,N}(\tfrac{1}{2}+it)|^2\,dt + \int_T^{2T} |\mathbf{D}_{k,M}(\tfrac{1}{2}+it)|^2\,dt.$$

where $MN = (T/2\pi)^k$ and $M, N \ge 1$. We can prove this when $k = 1$ or $k = 2$, provided that $M$ and $N$ are both $\ll T$. When $k \ge 3$, the known bounds for $E_k(s)$ are too large and it is difficult to show that the cross term really is small. (However, it might be possible to overcome these problems by appealing to a more complicated form of the approximate functional equation first developed by A. Good.)

Our problem now is to determine an asymptotic estimate for the mean square of the Dirichlet polynomials $\mathbf{D}_{k,N}(\frac{1}{2}+it)$ and $\mathbf{D}_{k,M}(\frac{1}{2}+it)$.

Montgomery and Vaughan's mean value theorem, Theorem 4.1, gives

$$\int_T^{2T} \left|\mathbf{D}_{k,N}(\tfrac{1}{2}+it)\right|^2 dt = \sum_{n \le N} \frac{d_k(n)^2}{n}(T + O(n))\,.$$

By standard techniques one can show that

$$\sum_{n \le N} d_k(n)^2 \sim \frac{a_k}{\Gamma(k^2)} N \log^{k^2-1} N$$

and that

$$\sum_{n \le N} \frac{d_k(n)^2}{n} \sim \frac{a_k}{\Gamma(k^2+1)} \log^{k^2} N\,,$$

where

$$a_k = \prod_p \left( \left(1 - \frac{1}{p}\right)^{k^2} \sum_{r=0}^{\infty} \frac{d_k^2(p^r)}{p^r} \right) .$$

Thus, for $N \ll T$, we deduce that

$$\int_T^{2T} \left|\mathbf{D}_{k,N}(\tfrac{1}{2}+it)\right|^2 \, dt \sim \frac{a_k}{\Gamma(k^2+1)} T \log^{k^2} N .$$

Using this with $M, N \ll T$ and $MN = (T/2\pi)^k$, we obtain the classical estimates for $I_1(T)$ and $I_2(T)$.

If $k \geq 3$, the condition $MN = (T/2\pi)^k$ forces at least one of $M$ or $N$ to be $\gg T$, so we need a mean value theorem for long Dirichlet polynomials. This requires good uniform estimates for the additive divisor sums

$$D_k(x,h) = \sum_{n \leq x} d_k(n) d_k(n+h) .$$

no such formula has been proved when $k > 2$, but a precise formula for the main term of $D_k(x,h)$ can be conjectured by a heuristic application of the circle method. This leads us to guess that

$$D_k(x,h) = m_k(x,h) + O(x^{1/2+\epsilon})$$

uniformly for $1 \leq h \leq x^{1/2}$, where $m_k(x,h)$ is a certain smooth function of $x$. Using this in the long mean value theorem, we obtain the

**Conjecture 1.** *Let $N = (T/2\pi)^{1+\eta}$ with $0 \leq \eta \leq 1$. Then*

$$\int_T^{2T} \left|\mathbf{D}_{k,N}(\tfrac{1}{2}+it)\right|^2 \, dt \sim w_k(\eta) \frac{a_k}{\Gamma(k^2+1)} T L^{k^2} ,$$

*where $a_k$ is the product over primes defined previously, and*

$$w_k(\eta) = (1+\eta)^{k^2} \left( 1 - \sum_{n=0}^{k^2-1} \binom{k^2}{n+1} \gamma_k(n) \left(1 - (1+\eta)^{-(n+1)}\right) \right) ,$$

*where*

$$\gamma_k(n) = (-1)^n \sum_{i=0}^{k} \sum_{j=0}^{k} \binom{k}{i} \binom{k}{j} \binom{n-1}{i-1, j-1, n-i-j+1}$$

*for $n \geq 1$, and$\gamma_k(0) = k$.*

The conjecture restricts us to $N \ll T^2$ . Thus, $M$ and $N$ in the formula

$$\int_T^{2T} |\zeta(\tfrac{1}{2}+it)|^{2k} \, dt \sim \int_T^{2T} |\mathbf{D}_{k,N}(\tfrac{1}{2}+it)|^2 \, dt + \int_T^{2T} |\mathbf{D}_{k,M}(\tfrac{1}{2}+it)|^2 \, dt.$$

must satisfy

$$M \ll T^2, \quad N \ll T^2, \quad \text{and} \quad MN = (T/2\pi)^k .$$

Writing $N = (T/2\pi)^{1+\eta}$ and $M = (T/2\pi)^{k-1-\eta}$, we find that

$$\int_T^{2T} |\zeta(\tfrac{1}{2}+it)|^{2k} \, dt \sim \int_T^{2T} |\mathbf{D}_{k,(T/2\pi)^{1+\eta}}(\tfrac{1}{2}+it)|^2 \, dt \quad + \int_T^{2T} |\mathbf{D}_{k,(T/2\pi)^{k-1-\eta}}(\tfrac{1}{2}+it)|^2 \, dt ,$$

where $0 \le \eta \le 1$. Thus,

$$\int_T^{2T} |\zeta(\tfrac{1}{2}+it)|^{2k}\, dt \sim \big(w_k(\eta) + w_k(k-2-\eta)\big)\frac{a_k}{\Gamma(k^2+1)} T L^{k^2}$$

**The 6th moment.** Take $k=3$. Then

$$\int_T^{2T} |\zeta(\tfrac{1}{2}+it)|^{6}\, dt \sim \big(w_3(\eta) + w_3(1-\eta)\big)\frac{a_3}{\Gamma(10)} T L^{9}$$

for $0 \le \eta \le 1$. We find from the conjecture that

$$w_3(\eta) = 1 + 9\eta + 36\eta^2 + 84\eta^3 + 126\eta^4 - 630\eta^5 + 588\eta^6 + 180\eta^7 - 9\eta^8 + 2\eta^9\,,$$

and one can verify that

$$w_3(\eta) + w_3(1-\eta) = 42$$

for $0 \le \eta \le 1$. Therefore

$$\int_T^{2T} |\zeta(\tfrac{1}{2}+it)|^{6}\, dt \sim 42\,\frac{a_3}{9!} T L^{9}\,.$$

**The 8th moment.** Take $k=4$. Then

$$\int_T^{2T} |\zeta(\tfrac{1}{2}+it)|^{8}\, dt \sim \big(w_4(\eta) + w_4(2-\eta)\big)\frac{a_4}{\Gamma(17)} T L^{16}\,,$$

where $\eta$ and $2-\eta$ must be in $[0,1]$. This forces $\eta = 1$. But

$$w_4(1) = 12012\,.$$

Hence

$$\int_T^{2T} |\zeta(\tfrac{1}{2}+it)|^{8}\, dt \sim 24024\frac{a_4}{16!} T L^{16}\,.$$

Originally we expected to be able to take $N > T^2$ in our formulas. That means we expected the error terms in

$$D_k(x,h) = m_k(x,h) + O_h(x^{1/2+\epsilon})$$

to behave independently when we average over $h$ up to $x^{1-\epsilon}$ (instead of just up to $x^{1/2-\epsilon}$). This is definitely not the case and we still do not understand why.