Modular forms and L-functions

David Farmer
AIM

Topics in classical automorphic forms (AMS)
by Iwaniec

Analytic Number Theory (AMS)
by Iwaniec & Kowalski

Books by Bump, Serre, Apostol, Koblitz

Website of Stein, Conrey-Farmer

L-functions:

- $L(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}$ Dirichlet Series

- $L(s) = \prod_p L_p(s)$
  where $L_p(s) = \sum_{j=0}^{\infty} \frac{a_{p^j}}{p^{js}}$ Euler Product

- $\xi_L(s) = Q^s \prod_{d=1}^{D} \Gamma\left(\frac{s}{2} + \mu_d\right) \cdot L(s)$ D = degree

  $= \varepsilon\, \overline{\xi}_L(1-s)$ Functional Equation

- $a_n \ll n^{\varepsilon}$ Ramanujan Bound

Rewrite the functional equation as

$$L(s) = \varepsilon\, \mathcal{X}(s)\, \overline{L}(1-s)$$

$$Z(t) = \left(\varepsilon \mathcal{X}\left(\tfrac{1}{2}+it\right)\right)^{-1/2} L\left(\tfrac{1}{2}+it\right)$$ real

$R_3$① $F_2$①

Dirichlet Series $\qquad$ Fourier Series

$$L(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s} \quad \xleftrightarrow{\text{mellin}} \quad f(z) = \sum_{n=1}^{\infty} a_n e^{2\pi i n z}$$

$$L(s) \Leftrightarrow L(1-s) \qquad\qquad f(z) = * f(z')$$

Functional Equation $\qquad$ Symmetry

Degree 0 $\qquad$ $L(s) \equiv 1$

Degree 1 $\qquad$ $\zeta(s)$ $\qquad$ $f(z) = \theta(z)$

$L(s, \chi)$ $\qquad$ $= \sum e^{2\pi i n^2 z}$

Degree 2 $\qquad$ $L(s, f)$ $\qquad$ f cusp form on $GL(2)$

(2 kinds of them)

also $\zeta(s) L(s, \chi)$

$F_2(3)$

$\mathcal{H} = \{ x + iy \in \mathbb{C} \mid y > 0 \}$

the "upper half plane"

$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{R})$

$\gamma : \mathcal{H} \to \mathcal{H}$

$$\gamma(z) = \frac{az+b}{cz+d}$$

$\Gamma \leq SL(2, \mathbb{R})$ discrete subgroup

$\mathcal{H}/\Gamma$ Riemann surface

Hecke congruence subgroups of $SL(2, \mathbb{Z})$

$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z}) : N / c \right\}$

N = level

$SL(2,\mathbb{R})$ acts on $\mathcal{H}$

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} : \mathcal{H} \to \mathcal{H}$$

$$\gamma z = \frac{az+b}{cz+d}$$

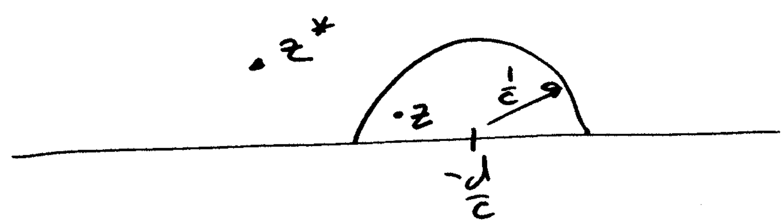
Check: $\quad \operatorname{Im}(\gamma z) = \dfrac{\operatorname{Im}(z)}{|cz+d|^2}$

Check: $\quad \gamma_1(\gamma_2 z) = (\gamma_1 \gamma_2)(z)$

Points in the "half disc" $\quad |cz+d| < 1$

ie $\quad |z + (-\frac{d}{c})| < \frac{1}{c}$

are moved "higher up"

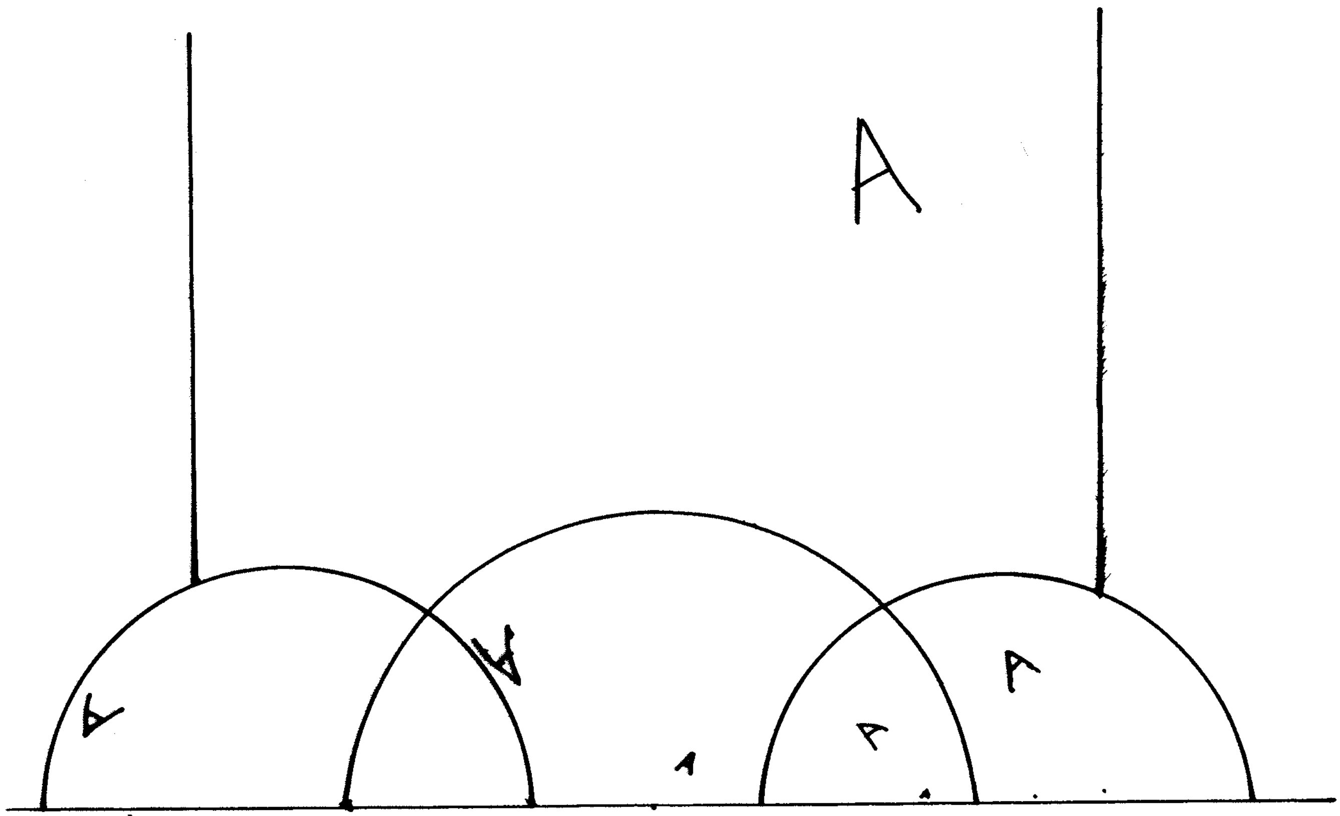


$\Gamma < SL(2,\mathbb{R})$ Discrete

A "highest point fundamental domain" for Γ is a collection of points as "high up as possible"

$\equiv$ Exterior of a collection of "half discs".

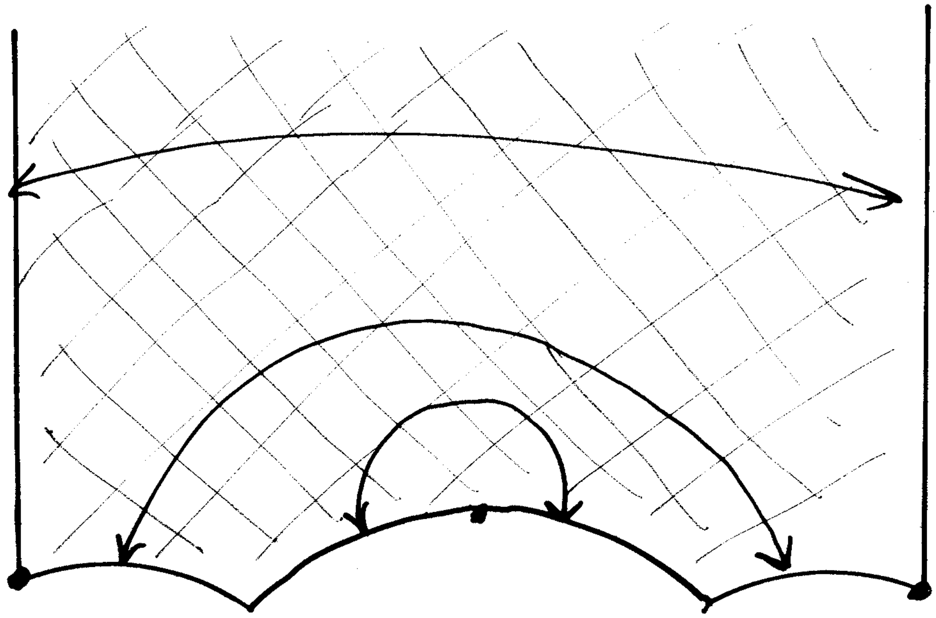
$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}(z) = z+1$



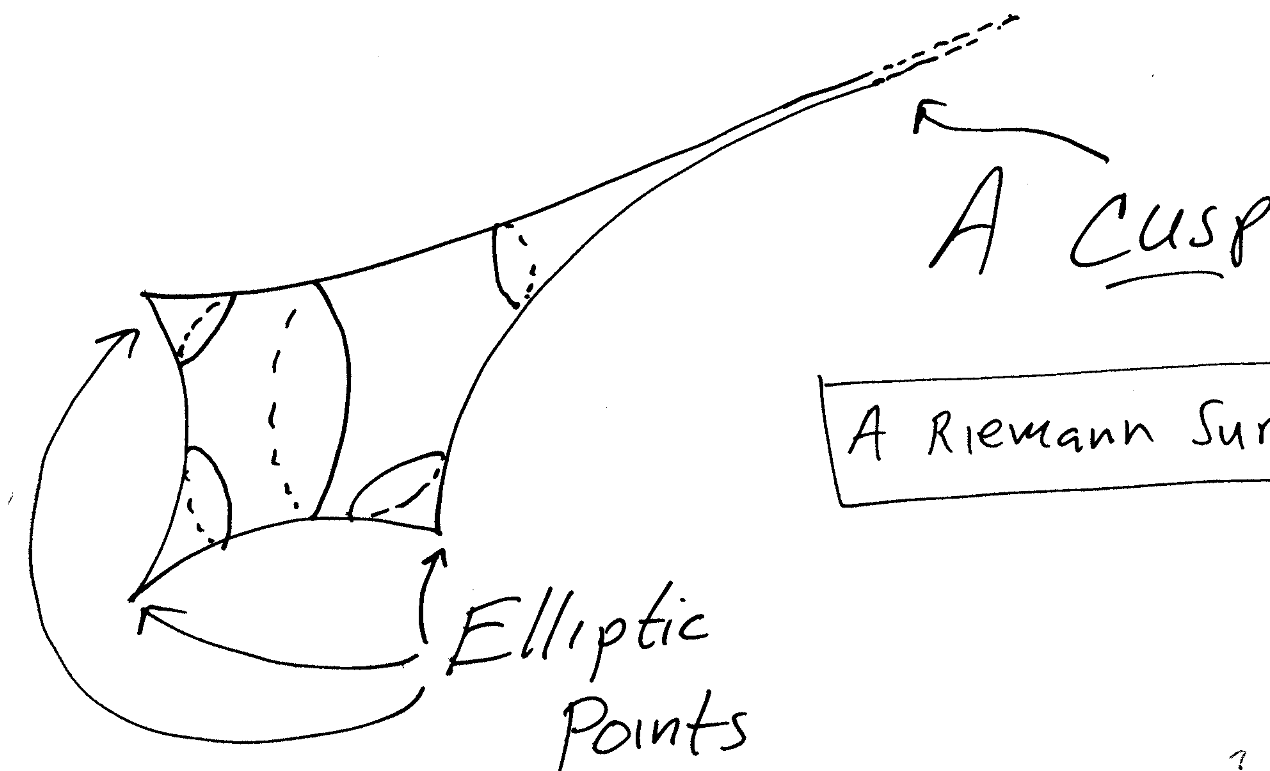
The A's are <u>all the same size</u>

The circles are straight lines.

This is "hyperbolic geometry."



FOLD UP THE FUNDAMENTAL DOMAIN



A Riemann Surface

Definition: A modular cusp form of weight k for the group Γ is a holomorphic function

$$f : \mathcal{H} \to \mathbb{C}$$

such that

$$f\left(\frac{az+b}{cz+d}\right) = (cz+d)^k f(z) \qquad \forall \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$$

which is bounded vanishes at all cusps of Γ.

$M_k(\Gamma)$	vector space of weight k modular forms for Γ
$S_k(\Gamma)$	" " " " k cusp forms for Γ

$$S_{k_1}(\Gamma_1)\, M_{k_2}(\Gamma_2) \subset S_{k_1+k_2}(\Gamma_1 \cap \Gamma_2)$$

Modular forms exist

$$\dim(S_k(\Gamma_0(N))) \approx \frac{kN}{12}$$ (there is a formula)

$\Gamma_0(1) = SL(2,\mathbb{Z})$ generated by $\begin{pmatrix}1&1\\0&1\end{pmatrix}(z) = z+1$

$\begin{pmatrix}0&-1\\1&0\end{pmatrix}(z) = \frac{-1}{z}$

Proof: Euclidean algorithm

Eisenstein series:

$$G_k(z) = \sum_{n,m}{}' \frac{1}{(mz+n)^k} \qquad k \geq 4, \text{ even}$$

$$\in M_k(\Gamma_0(1))$$

Exercise: Check the generators

G_k not a cusp form because

$$G_k(i\infty) = \sum_n{}' \frac{1}{n^k} = 2\zeta(k)$$

Theorem: G_4 and G_6 freely generate the ring of modular forms on $SL(2,\mathbb{Z})$.

Corollary: The first cusp form on $SL(2,\mathbb{Z})$ has weight 12

Linear operators on $S_k(\Gamma_0(N))$

$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{R})$

Notation: $$\left(f \big|_k \gamma\right)(z) = (cz+d)^{-k} f(\gamma z)$$

$$f\left(\frac{az+b}{cz+d}\right) = (cz+d)^k f(z) \iff f|\gamma = f$$

"stroke" or "slash"

The Fricke involution: $H_N = \begin{pmatrix} 0 & -1 \\ N & 0 \end{pmatrix}$

$$= \begin{pmatrix} 0 & -1/\sqrt{N} \\ \sqrt{N} & 0 \end{pmatrix}$$

$H_N : S_k(\Gamma_0(N)) \longrightarrow S_k(\Gamma_0(N))$,

so $S_k(\Gamma_0(N))$ has a basis of eigenfunctions of Fricke.

<u>Proof:</u> $(f|H_N)|\gamma = f|\underbrace{H_N \gamma H_N}_{} H_N = f|H_N$

because $$\begin{pmatrix} 0 & -1 \\ N & 0 \end{pmatrix}\begin{pmatrix} a & b \\ cN & d \end{pmatrix}\begin{pmatrix} 0 & -1 \\ N & 0 \end{pmatrix} = \begin{pmatrix} -d & c \\ bN & -a \end{pmatrix}$$

Exercise: Find all the white lies on this page

$f \in S_k(\Gamma_0(N))$, eigenfunction of Fricke.

$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in \Gamma_0(N)$, so $f(z) = f(z+1)$

Fourier expansion $\quad f(z) = \sum_{n=1}^{\infty} a_n e^{2\pi i n z}$

Note: $n \geq 1$.

Mellin transform to get an L-function:

$$\int_0^\infty f(iy)\, y^s \frac{dy}{y} = \int_0^\infty \sum_{n=1}^\infty a_n e^{-2\pi n y} y^s \frac{dy}{y}$$

Re(s) large enough
$$= \sum_{n=1}^\infty a_n \int_0^\infty e^{-2\pi n y} y^s \frac{dy}{y}$$

$y \to \frac{y'}{2\pi n}$
$$= (2\pi)^{-s} \sum_{n=1}^\infty \frac{a_n}{n^s} \int_0^\infty e^{-y} y^s \frac{dy}{y}$$

$$= (2\pi)^{-s} L(s)\, \Gamma(s)$$

R_3''

Functional equation

$$(2\pi)^{-s}\,\Gamma(s)\,L(s) = \int_0^\infty f(iy)\, y^s \frac{dy}{y}$$

$$f|H_N = \pm f$$

$$f\left(\frac{-1}{Nz}\right) = \pm N^{k/2} z^k f(z)$$

$$f\left(\frac{i}{Ny}\right) = \pm N^{k/2} i^k y^k f(iy)$$

Hecke's trick

$$= \int_A^\infty f(iy)\, y^s \frac{dy}{y} + \int_0^A f(iy)\, y^s \frac{dy}{y}$$

$$= \quad '' \quad \pm \frac{1}{N^{k/2} i^k} \int_0^A f\left(\frac{i}{Ny}\right) y^{s-k} \frac{dy}{y}$$

$y \to \frac{1}{Ny'}$

$$= \int_A^\infty f(iy)\, y^s \frac{dy}{y} \pm \frac{N^{k-s}}{N^{k/2} i^k} \int_{\frac{1}{NA}}^\infty f(iy)\, y^{k-s} \frac{dy}{y}$$

RHS entire

Choose $A = 1/\sqrt{N}$ to get

$$\Lambda(s) = \left(\frac{\sqrt{N}}{2\pi}\right)^s \Gamma(s)\, L(s) = \pm i^k \Lambda(k-s)$$

Hecke operators

$$T_p = \begin{pmatrix} p & \\ & 1 \end{pmatrix} + \begin{pmatrix} 1 & \\ & p \end{pmatrix} + \begin{pmatrix} 1 & 1 \\ & p \end{pmatrix} + \cdots + \begin{pmatrix} 1 & p-1 \\ & p \end{pmatrix}$$

$$T_p : S_k(\Gamma_0(N)) \to S_k(\Gamma_0(N)) \qquad \text{if } p \nmid N$$

(other operators for $p \mid N$)

Demonstration: $T_2 : S_k(\Gamma_0(1)) \to S_k(\Gamma_0(1))$.

$$T_2 \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \left(\begin{pmatrix} 2 & \\ & 1 \end{pmatrix} + \begin{pmatrix} 1 & \\ & 2 \end{pmatrix} + \begin{pmatrix} 1 & 1 \\ & 2 \end{pmatrix} \right) \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 2 & 2 \\ & 1 \end{pmatrix} + \begin{pmatrix} 1 & 1 \\ & 2 \end{pmatrix} + \begin{pmatrix} 1 & 2 \\ 0 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 2 \\ & 1 \end{pmatrix} \begin{pmatrix} 2 & \\ & 1 \end{pmatrix} + \begin{pmatrix} 1 & 1 \\ & 2 \end{pmatrix} + \begin{pmatrix} 1 & 1 \\ & 1 \end{pmatrix} \begin{pmatrix} 1 & \\ & 2 \end{pmatrix}$$

$$\equiv \begin{pmatrix} 2 & \\ & 1 \end{pmatrix} + \begin{pmatrix} 1 & 1 \\ & 2 \end{pmatrix} + \begin{pmatrix} 1 & \\ & 2 \end{pmatrix}$$

$$= T_2$$

ie, $(f \mid T_2) \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = f \mid T_2$

Exercise: Do the same for $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

The T_p commute with each other and with Fricke, so we can choose a basis of $S_k(\Gamma_0(N))$ of simultaneous eigenfunctions

$f | T_p = c_p f$ means what?

$$f(z) = \sum_{n=1}^{\infty} a_n e^{2\pi i n z} = \sum_{n=1}^{\infty} a_n e(nz) \qquad \boxed{a_1 = 1}$$

$$f | T_2 = f(2z) + f(z/2) + f\left(\tfrac{z+1}{2}\right)$$

$$= e(2z) + a_2 e(4z) + a_3 e(6z) + \cdots$$

$$+ e(z/2) + a_2 e(z) + a_3 e(\tfrac{3}{2}z) + a_4 e(2z) + \cdots$$

$$+ e(\tfrac{z}{2}) e(\tfrac{1}{2}) + a_2 e(z) e(1) + a_3 e(\tfrac{3z}{2}) e(\tfrac{3}{2}) + \cdots$$

$$= a_2 e(z) + (a_4 + 1) e(2z) + a_6 e(3z) + \cdots$$

$$= c_2 (e(z) + a_2 e(2z) + a_3 e(3z) + \cdots)$$

iff $\quad c_2 = a_2, \quad a_2^2 = a_4 + 1, \quad a_2 a_3 = a_6$, etc

Note: I was sloppy with the weight k.

P_3^{14}

f eigenfunction of Hecke operators

$$\Updownarrow$$

$L(s,f)$ has a particular Euler product
(Chris Hughes' homework prior to school).

$f \in H_k(N)$ = Hecke basis of $S_k(\Gamma_0(N))$

$$f(z) = \sum_{n=1}^{\infty} a_n e(nz)$$

$$L(s,f) = \sum_{n=1}^{\infty} \frac{a_n n^{\frac{k-1}{2}}}{n^s} \qquad L(s) \Leftrightarrow L(1-s)$$

<u>Moments</u>: $\displaystyle\sum_{f \in H_k(N)} L(\tfrac{1}{2}, f)^M$ as $k \to \infty$ or $N \to \infty$

<u>Zeros</u>: Distribution of low-lying zeros (small imaginary part) $\qquad //$