Michael Rubinstein, lecture 2

- Petersson formula
- Selberg Trace formula (for Hecke operators)
- Applications to moments of L-functions

David talked about cusp forms

for

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma(1) \,\middle|\, c \equiv 0 \bmod N \right\}$$

where

$$\Gamma(1) = SL_2(\mathbb{Z}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \,\middle|\, \begin{matrix} a,b,c,d \in \mathbb{Z} \\ ad-bc=1 \end{matrix} \right\}$$

In my talk I'll focus just on $\Gamma(1)$,

but all formulas presented generalize

to $\Gamma_0(N)$.

Cusp forms

A holomorphic function on $\mathbb{H}$ is said to be a cusp form for $\Gamma(1)$ of weight $k \in \mathbb{Z}$, denoted $f \in S_k(\Gamma(1))$ if:

1) $f\left(\frac{az+b}{cz+d}\right) = (cz+d)^k f(z)$ for all $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma(1)$.

2) $f(it) \to 0$ as $t \to \infty$

If we replace 2) by $f(it)$ is bounded as $t \to \infty$ then f is called a modular form of weight k for $\Gamma(1)$ and is denoted by $f \in M_k(\Gamma(1))$.

ex/ Show k must be positive, even

$\Gamma(1)$ acts on $\mathbb{H}$ by linear fractional transformations, $\mathbb{H}$ stands for the upper half plane.

If $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma(1)$

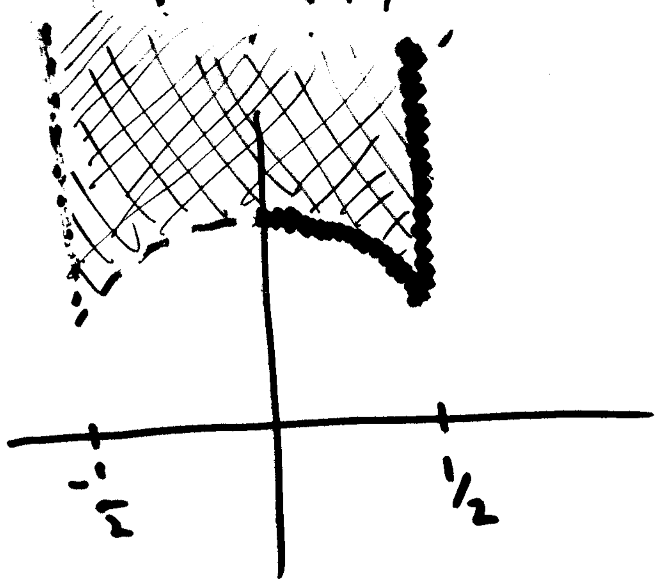
then

defn: $\gamma z = \dfrac{az+b}{cz+d}$.

check:

$\mathrm{Im}\left(\dfrac{az+b}{cz+d}\right) = \dfrac{\mathrm{Im}\, z}{|cz+d|^2}$ so γz maps $\mathbb{H}$ to $\mathbb{H}$

This partitions $\mathbb{H}$ into equivalence classes $\Gamma(1)\backslash\mathbb{H}$



Two ways to get L-functions

out of $\Gamma(1)$

1) cusp forms - certain holomorphic functions on $\Gamma(1)\backslash\mathbb{H}$.
2) Maass forms - certain non-holomorphic functions on $\Gamma(1)\backslash\mathbb{H}$

Why this leads to a rich theory:

- Tools from harmonic analyis
- symmetries inherent in the action of $\Gamma(1)$ allows us to turn cusp/maass forms into functions that have a similar functional equation to $\xi(s)$
- existence of certain operators, the Hecke operators, that give rise to cusp/maass forms with remarkable multiplicative properties, which allows us to mimic classic theory of primes.

$f \in S_k(\Gamma(1)) \to$

- $f(z+1) = f(z)$

$$f(z) = \sum_{1}^{\infty} a_n e^{2\pi i n z}$$

Let

$$L_f(s) = \sum_{1}^{\infty} \frac{a_n}{n^{\frac{k-1}{2}}} \cdot \frac{1}{n^s}$$

- $f(-1/z) = z^k f(z) \longrightarrow$ mimic Riemann's proof of functn'l eqn for $\xi(s)$

$$\int_0^\infty = \int_0^1 + \int_1^\infty$$

$$\Lambda_f(s) = (2\pi)^{-s} \Gamma\left(s + \frac{k-1}{2}\right) L_f(s)$$

$$= i^k \Lambda_f(1-s)$$

$S_k(\Gamma(1))$ is a finite dimensional vector space over $\mathbb{C}$.

pf

'vector space': linear combinations of cusp forms is a cusp form.

'finite dimensional':

- for $\Gamma(1)$ we can give a very direct proof (using Ramanujan Δ and G_k, Eisenstein series).
- we'll see a proof using Selberg trace formula (generalizes to $\Gamma_0(N)$)
- can also use theory of Riemann surfaces (works for $\Gamma_0(N)$).

Remark: 'finite dimensional' leads to all sorts of incredible arithmetic identities amongst coefficients of cusp forms

Hecke operators $T(n)$ on $S_k(\Gamma(1))$

Defn

$$(T(n)f)(z) = n^{k-1} \sum_{ad=n} d^{-k} \sum_{b \bmod d} f\left(\frac{az+b}{d}\right)$$

$$T(n) : S_k(\Gamma(1)) \to S_k(\Gamma(1))$$

- linear transformation of finite dimensional vectorspace

- self adjoint with respect to Petersson inner product

Petersson inner product $\quad \langle f, g \rangle = \int f \bar{g} \, y^k \frac{dx\,dy}{y^2}$

$$\langle T_n f, g \rangle = \langle f, T_n g \rangle$$

1) $T_n T_m = \sum_{d \mid (n,m)} d^{k-1} T_{\frac{nm}{d^2}}$

2) If $f(z) = \sum_{1}^{\infty} a(m) e(mz)$

then

$$T(n) f(z) = \sum_{1}^{\infty} b(m) e(mz)$$

where

$$b(m) = \sum_{d \mid (n,m)} d^{k-1} a\left(\frac{mn}{d^2}\right)$$

If f is an eigenvector of all the $T(n)$:

$$T(n) f(z) = \underbrace{\lambda(n) n^{\frac{k-1}{2}}}_{\text{how we prefer to write the eigenvalues}} f(z)$$

for all $n \geq 1$.

Thus

$$\lambda(n)\, n^{\frac{k-1}{2}}\, a(m) = \sum_{d \mid (n,m)} d^{k-1} a\left(\frac{nm}{d^2}\right)$$

put $m=1$, assume $a(1)=1$ (if not, normalize f so it is)

So

$$\lambda(n)\, n^{\frac{k-1}{2}} = a(n)$$

i.e. fourier coefficients $a(n)$ are just the Hecke eigenvalues

and hence

$$a(n)\,a(m) = \sum_{d \mid (n,m)} d^{k-1} a\left(\frac{nm}{d^2}\right)$$

$\longrightarrow$ $a(n)a(m) = a(nm)$ if $(n,m)=1$

$$a(p^{r+1}) = a(p)\,a(p^r) - p^{k-1} a(p^{r-1}), \quad r \geq 1$$

p prime

The existence of a basis for $S_k(\Gamma(1))$ of simultaneous eigenvectors for all the $T(n)$:

- selfadjointness gives us eigenvectors for each $T(n)$
- $T(n)$'s commute gives us simultaneous eigenvectors

ex/ $\dim S_{12}(\Gamma(1)) = 1$

Let $z \in \mathbb{H}$, $q = e^{2\pi i z}$

$$\Delta(z) = q \prod_{1}^{\infty} (1-q^n)^{24} = \sum_{1}^{\infty} \tau(n) q^n$$

called Ramanujan tau function

is called the Ramanujan-delta cusp form of weight 12 (prove via transformation property of 'eta' function).
uses, as usual, poisson summation

Ramanujan noticed

- $\tau(mn) = \tau(m)\tau(n)$ if $(m,n)=1$
- $\tau(p^{n+1}) = \tau(p^n)\tau(p) - p^{11}\tau(p^{n-1})$

which gives

$$\sum_{1}^{\infty} \frac{\tau(n)}{n^{11/2}} \cdot \frac{1}{n^s} = \prod_{p} \left(1 - \frac{\tau(p)}{p^{11/2}} p^{-s} + p^{-2s}\right)^{-1}$$

Since $\dim S_{12}(\Gamma(1)) = 1$, $\Delta(z)$ is automatically a Hecke eigenform, so $\tau(n)$'s satisfy multiplicative properties noticed by Ramanujan. The same idea applies to $k = 16, 18, 20, 22$ since $\dim S_k(\Gamma(1)) = 1$ for these k.

$\dim S_{14}(\Gamma(1)) = 0$

One can find explicitly a basis for $S_k(\Gamma(1))$ using Eisenstein series (certain modular forms of weight k), or eisenstein series in combination with $\Delta(z)$.

One can then find a basis for $S_k(\Gamma(1))$ consisting of Hecke eigenforms by diagonalizing say $T(2)$ relative to this basis.

The space of cusp forms of weight k for $\Gamma(1)$,

$S_k(\Gamma(1))$, is spanned by

$$\Delta^l G_{k-12l}, \quad 1 \le l \le \frac{k-4}{12}$$

together with

$$\Delta^{k/12} \quad \text{if } 12 \mid k$$

where $\Delta(z)$ is the Ramanujan delta function

and

$$G_k(z) = \frac{(k-1)!}{2(2\pi i)^k} \sum{}' (mz+n)^{-k}$$

$$= -\frac{B_k}{2k} + \sum_1^\infty \sigma_{k-1}(n) q^n, \quad q = e^{2\pi i z}$$

Bernoulli numbers

divisor function

$$\sum_{d \mid n} d^{k-1}$$

assuming formula for $\dim S_k$.

pf We have the correct number of cusp forms, linearly independent over $\mathbb{C}$ (consider leading coefficient). A direct pf not using $\dim S_k$ formula is possible.

$$S(m,n;c) = \sum_{a\bar{a}\equiv 1 (c)} e\left(\frac{m\bar{a}+na}{c}\right), \quad e(z)=e^{2\pi i z}$$

is a Kloosterman sum.

What is this formula good for?

When analyzing moments of L-functions $L_f(s)$ associated to $f \in H_k$, averaged w.r.t. Petersson weighting, one needs estimates for averages of products of Dirichlet coefficients.

By multiplicative relations, expressions of the form

$$\lambda_f(m_1) \cdots \lambda_f(m_k)$$

can be written as

$$\sum_{\substack{j \geq 0 \\ p}} b_j \lambda_f(p^j)$$

(use $\lambda_f(m)\lambda_f(n) = \sum_{d|(m,n)} \lambda_f(mn/d^2)$)

Now

Weil:

$$|S(m,n;c)| \leq (m,n,c)^{1/2} c^{1/2} \sigma_0(c)$$

divisors of c

and using series expansion for

$$J_{k-1}(x)$$

hmule the l.h.s $\to \delta(m,n)$ as $k \to \infty$.

Applying this with $m=1$, we get, as $k \to \infty$

$$\frac{\Gamma(k-1)}{(4\pi)^{k-1}} \sum_{f \in H_k} \frac{\lambda_f(n)}{\langle f,f \rangle} \to \begin{cases} 1 & \text{if } n=1 \\ 0 & \text{otherwise} \end{cases}$$

and

$$H(m) = \begin{cases} 0 & \text{if } m<0 \\ -1/12 & \text{if } m=0 \end{cases}$$

\# equivalence classes w.r.t. $\Gamma(1)$ if $m>0$ of quadratic forms $ax^2+bxy+cy^2$ with $b^2-4ac=-m$, counting forms equivalent to a multiple of x^2+y^2 (resp x^2+xy+y^2) with weight $\frac{1}{2}$ (resp $1/3$).

(more or less the class numbers of imag quad fields)

table: $H(m)=0$ if $m \equiv 1$ or $2 \bmod 4$

m	$H(m)$
0	$-1/12$
3	$1/3$
4	$1/2$
7	1
8	1
11	1
12	$4/3$
15	2
16	$3/2$

Petersson Formula

Let $H_k = \{ f_1, \dots f_{\dim S_k} \}$ be eigenfunctions of the Hecke operators (normalized so that leading coefficient is 1).

The Petersson formula gives orthogonality relations for the coefficients of $f \in H_k$, but weighted according to Petersson norm

$$\frac{\Gamma(k-1)}{(4\pi)^{k-1}} \sum_{f \in H_k} \frac{\lambda_f(m)\lambda_f(n)}{\langle f, f \rangle} = \delta(m,n) + 2\pi i^{-k} \sum_{1}^{\infty} \frac{S(m,n;c)}{c} J_{k-1}\left(\frac{4\pi\sqrt{mn}}{c}\right)$$

J_ν - Bessel function

$$J_\nu(x) = \sum_{0}^{\infty} \frac{(-1)^m}{m!\,\Gamma(m+\nu+1)} \left(\frac{x}{2}\right)^{2m+\nu}$$

Another useful formula for computing averages according to usual weight:

Selberg Trace formula for $T(n)$.

$$\mathrm{tr}(T(n)) = \sum_{f \in H_k} \lambda_f(n)$$

$$= \frac{-1}{2} \sum_{m=-\infty}^{\infty} \frac{\alpha_m^{k-1} - \bar{\alpha}_m^{k-1}}{\alpha_m - \bar{\alpha}_m} H(4n - m^2)$$

$$-\frac{1}{2} \sum_{dd'=n} \min(d, d')^{k-1}$$

where

$\alpha_m, \bar{\alpha}_m$ roots of $x^2 - mx + n$

$\frac{\alpha_m^{k-1} - \bar{\alpha}_m^{k-1}}{\alpha_m - \bar{\alpha}_m}$ coeff of x^{k-2} of $\frac{1}{1 - mx + nx^2}$

Applications

1) Take $n=1$, then $\lambda_f(1)=1$, so $tr(T_1) = \dim S_k$

r.h.s of trace formula is a finite sum.

Shows $\dim S_k(\Gamma(1))$ is finite, and in fact

equals the coefficient of x^{k-2} in:

$$-\frac{1}{2}\left(\frac{H(4)}{1+x^2} + \frac{H(3)}{1-x+x^2} + \frac{H(3)}{1+x+x^2} + \frac{H(0)}{1-2x+x^2} + \frac{H(0)}{1+2x+x^2} \right)$$

$$-\frac{1}{2} \cdot \frac{1}{1-x^2}$$

Now coefficients of $\frac{1}{1+x^2}, \frac{1}{1-x+x^2}, \frac{1}{1+x+x^2}, \frac{1}{1-x^2}$

are all ± 1, and

$$\frac{1}{1+2x+x^2} = 1-2x+3x^2-4x^3+\cdots \quad \text{and } H(0) = -1/12$$

$$\frac{1}{1-2x+x^2} = 1+2x+3x^2+4x^3+\cdots$$

so $\dim S_k = \frac{k}{12} + O(1)$

2) The Selberg Trace formula for $T(n)$

also gives the average of $\lambda_f(n)$ over $f \in H_k$:

$$\frac{12}{k} \sum_{f \in H_k} \lambda_f(n) \sim n^{-1/2}$$

Can be used in combination with mult properties of λ to study moments of $\lambda_f(p)$ for fixed p varying f over H_k.

Sarnak, Serre
Conrey - Duke - Farmer

If we write

$\lambda_f(p) = 2\cos\theta_f(p)$ (it is known, by Deligne, that $\lambda_f(p) \in [-2,2]$)

then

$\theta_f(p)$ becomes as $k \to \infty$, uniformly distributed

w.r.t. to the measure

$$\frac{2}{\pi}\left(1+\frac{1}{p}\right)\frac{\sin^2\theta}{\left(1-\frac{1}{p}\right)^2+\frac{4}{p}\sin^2\theta}\,d\theta \to \frac{2}{\pi}\sin^2\theta\,d\theta \quad \left(\text{the Sato-Tate measure}\right)$$

Amongst all $\begin{pmatrix} * & * \\ c & d \end{pmatrix} \in \Gamma(1)$ select just one matrix.

so $(c,d)=1$

1) $P_m(z,k) \in S_k(\Gamma(1))$

2) $P_m(z,k)$ span $S_k(\Gamma(1))$, $m=1,2,\dots$

i.e amongst the P_m's we can find a basis for $S_k(\Gamma(1))$.

so, estimates for fourier coefficients of $P_m(z,k)$ gives us estimates for coefficients of any $f \in S_k(\Gamma(1))$. Generalizes to $\Gamma_0(N)$ where 'Ramanujan conjecture' is not proven.

<u>pf 2)</u> Let $f \in S_k(\Gamma(1))$, $f(z) = \sum_1^\infty a_n e(nz)$

Show $\langle P_m, f \rangle = \bar{a}_m \dfrac{\Gamma(k-1)}{(4\pi m)^{k-1}}$, hence

there is no $f \in S_k(\Gamma(1))$ which is simultaneously orthogonal to all the P_m's. Thus orthogonal complement of span($\{P_m\}$) is just the zero vector.

Outline of how to get Petersson Formula

Poincaré series

Let

$$P_m(z,k) = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma(1)} (cz+d)^{-k} e(m\gamma z)$$

$\Gamma_\infty \backslash \Gamma(1)$ coset representatives

where $\Gamma_\infty = \left\{ \begin{pmatrix} 1 & r \\ 0 & 1 \end{pmatrix} \middle| r \in \mathbb{Z} \right\}$

Two matrices in $\Gamma(1)$, $\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}$

give rise to the same coset if $\exists r_1, r_2 \in \mathbb{Z}$:

$$\begin{pmatrix} 1 & r_1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & r_2 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}$$

$\longrightarrow$ $c = c'$, $d = d'$

Also, $\Gamma(1)$ requires $ad - bc = 1 \rightarrow (c,d) = 1$

examining top two entries gives representatives...

One can also compute

$$P_m(z,k) = \sum_{1}^{\infty} \hat{P}_m(n)\, e(nz)$$

where

$$\hat{P}_m(n) = \left(\frac{n}{m}\right)^{\frac{k-1}{2}} \left(\delta(m,n) + 2\pi i^{-k} \sum_{c>0} J_{k-1}\left(\frac{4\pi\sqrt{mn}}{c}\right) \frac{S(m,n;c)}{c} \right)$$

To get
Petersson formula:

Let $H_k = \{f_1, \dots f_{\dim S_k}\}$ basis for $S_k(\Gamma(1))$ consisting of eigenvectors of all the Hecke operators. write

$$P_m(z,k) = \sum_{1}^{\dim S_k} \alpha_j f_j(z)$$

f_j's are mutually orthogonal w.r.t. Petersson inner product.

So,

$$\langle P_m(z,k), f_j(z) \rangle = \alpha_j \langle f_j, f_j \rangle$$

writing

$$f_j(z) = \sum_1^\infty a_f(n) e(nz) = \sum_1^\infty \lambda_f(n) n^{\frac{k-1}{2}} e(nz)$$

then, by above,

$$\langle P_m(z,k), f \rangle = \lambda(m) m^{\frac{k-1}{2}} \frac{\Gamma(k-1)}{(4\pi m)^{k-1}}$$

so

$$\alpha_j = \lambda_{f_j}(m) m^{\frac{k-1}{2}} \frac{\Gamma(k-1)}{(4\pi m)^{k-1}} \cdot \frac{1}{\langle f_j, f_j \rangle}$$

$$P_m(z,k) = \frac{\Gamma(k-1) m^{\frac{k-1}{2}}}{(4\pi m)^{k-1}} \sum_{f \in H_k} \lambda_f(m) \frac{f(z)}{\langle f, f \rangle}$$

Comparing coefficient of $e(nz)$ on both sides gives Petersson formula.