Orthogonal polynomials techniques

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We want to compute the correlation functions of a point process on an interval $J \subseteq \mathbb{R}$ whose j.p.d.f. is of the form

$$\boxed{P_{N2}(\lambda_1, \ldots, \lambda_N) = C \prod_{j=1}^{N} w(\lambda_j) \prod_{1 \le j < k \le N} \left|\lambda_k - \lambda_j\right|^2 ,}$$

where $w(\lambda) > 0$ is a real function such that the integral

$$\int_J w(\lambda)\lambda^k \mathrm{d}\lambda$$

is finite for any k. $w(\lambda)$ ***is called weighting function***.

Classical $\beta = 2$ ensembles.

Classical weighting functions:

$$w(\lambda) = \begin{cases} e^{-\lambda^2} & \text{Hermite} \\ \lambda^a e^{-\lambda} \quad (\lambda > 0) & \text{Laguerre} \\ (1-\lambda)^a (1+\lambda)^b \quad (-1 < \lambda < 1) & \text{Jacobi} \\ \left(1+\lambda^2\right)^{-a} & \text{Cauchy} \end{cases}$$

Each of these weighting functions generate a family $\{p_j(\lambda)\}$ of classical orthogonal polynomials.

Question: *Which are the matrix ensembles whose spectral correlation functions can be computed using these polynomials?*

1. Hermite polynomials $\longrightarrow$ GUE ensemble, i.e. matrices $H = (h_{jk})$ such that
$$h_{jk} = \overline{h}_{kj}$$
whose independent elements are i.i.d. normal random variables.

2. Laguerre polynomials $\longrightarrow$ Laguerre ensemble. They describe the j.p.d.f. of an ensemble of *Wishart* matrices, i.e. matrices of the type
$$A = XX^*,$$
where the elements of X are complex normal random variables. With applications to statistics (study of covariance matrices) and engineering.

3. Jacobi polynomials $\longrightarrow$ j.p.d.f. of the classical compact groups $\mathrm{SO}^{\pm}(N)$ and $\mathrm{Sp}(2N)$.

4. Cauchy polynomials $\longrightarrow$ certain types of circular ensembles.

Haar measure of the classical compact groups (except U(N)) is

$$P^N_{(\sigma_1,\sigma_2)}(\theta_1,\ldots,\theta_N) = \frac{1}{Z_N^{(\sigma_1,\sigma_2)}}\prod_{l=1}^{N}(1+\cos\theta_l)^{\sigma_1+1/2}(1-\cos\theta_l)^{\sigma_2+1/2} \times \prod_{1\le j<k\le N}(\cos\theta_j-\cos\theta_k)^2,$$

$$(\sigma_1,\sigma_2) = \begin{cases} (-1/2,-1/2) & \mathrm{SO}^+(2N) \\ (1/2,1/2) & \mathrm{Sp}(2N) \quad \text{and} \quad \mathrm{SO}^-(2N+2) \\ (-1/2,1/2) & \mathrm{SO}^+(2N+1) \\ (1/2,-1/2) & \mathrm{SO}^-(2N+1) \end{cases}$$

By setting $x = \cos\theta$, we obtain

$$\boxed{P_{N2}(x_1, \ldots, x_N) = C \prod_{j=1}^{N} (1 + x_j)^{\sigma_1} (1 - x_j)^{\sigma_2} \prod_{1 \le j < k \le N} (x_j - x_k)^2}$$

The orthogonal polynomials $\{p_j^{(\sigma_1,\sigma_2)}(x)\}$ that will arise are are called *Jacobi polynomials* and are orthogonal in $-1 \le x \le 1$ with respect to the weight

$$\boxed{w_{(\sigma_1,\sigma_2)}(x) = (1 + x_j)^{\sigma_1} (1 - x_j)^{\sigma_2} \,,}$$

that is

$$\int_{-1}^{1} w_{(\sigma_1,\sigma_2)}(x) p_j^{(\sigma_1,\sigma_2)}(x) p_k^{(\sigma_1,\sigma_2)}(x) dx = \delta_{jk}.$$

Computing the spectral correlation for U(N), i.e.

$$\boxed{P_{N2}(\theta_1,\ldots,\theta_N)=\frac{1}{(2\pi)^N N!}\prod_{1\le j<k\le N}\left|e^{i\theta_k}-e^{i\theta_j}\right|^2.}$$

works with the same ideas.

The orthogonal polynomials are $\phi_j(z)=z^j$, with $z=e^{i\theta}$:

$$\frac{1}{2\pi}\int_0^{2\pi} f(\theta)\overline{\phi_j(z)}\phi_k(z)d\theta=\delta_{jk},$$

with $f(\theta)=1$.

Orthogonal polynomials

Theorem. *Given an arbitrary weighting function* $w(\lambda)$ *there exists a unique (up to a constant of proportionality) sequence of polynomials* $p_k(\lambda)$ *such that*

$$(p_j, p_k) = \int_J w(\lambda) p_j(\lambda) p_k(\lambda) \mathrm{d}\lambda = N_k \delta_{jk}$$

- The sequence $p_j(\lambda)$ are called ***orthogonal polynomials*** with weight $w(\lambda)$.
- We will fix the arbitrary constant of proportionality by setting the coefficient of x_k in $p_k(\lambda)$ equal to one. That is, we consider ***monic polynomials.***

Our first goal is to proof

Theorem 1. *Let $p_j(\lambda)$ be a set of orthogonal monic polynomials with weight $w(\lambda)$. The following identity holds*

$$\prod_{l=1}^{N} w(\lambda_l) \prod_{1\le j<k\le N} |\lambda_k - \lambda_j|^2 = \left(\prod_{l=0}^{N-1} (p_l, p_l)\right) \det_{N\times N} \left(K_N(\lambda_j, \lambda_k)\right),$$

where

$$K_N(\lambda, \mu) = (w(\lambda)w(\mu))^{1/2} \sum_{j=0}^{N-1} \frac{p_j(\lambda)p_j(\mu)}{(p_j, p_j)}.$$

and

$$R_n(\lambda_1, \dots, \lambda_n) = \det_{n\times n}\big(K_N(\lambda_j, \lambda_k)\big),$$

where

$$R_n(\lambda_1,\ldots,\lambda_n)=\frac{N!}{(N-n)!}\int_J\cdots\int_J P_{N2}(\lambda_1,\ldots,\lambda_N)d\lambda_{n+1}\cdots d\lambda_N$$

Remember: the spectral correlations of random matrices in U(N) had exactly this structure with

$$\boxed{K_N(\theta,\phi)=\frac{1}{2\pi}\frac{\sin\left(\frac{N(\phi-\theta)}{2}\right)}{\sin\left(\frac{\phi-\theta}{2}\right)}}$$

Notation:

We denote by $(p_j(\lambda_k))$ the matrix

$$\begin{pmatrix} p_0(\lambda_1) & p_0(\lambda_2) & \cdots & p_0(\lambda_n) \\ p_1(\lambda_1) & p_1(\lambda_2) & \cdots & p_1(\lambda_n) \\ p_2(\lambda_1) & p_2(\lambda_2) & \cdots & p_2(\lambda_n) \\ \hdotsfor{4} \\ p_{n-1}(\lambda_1) & p_{n-1}(\lambda_2) & \cdots & p_{n-1}(\lambda_N) \end{pmatrix},$$

where $p_j(\lambda)$ is a polynomial of degree j.

The choice $p_j(\lambda) = \lambda^j$ gives

$$V^t = \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ \lambda_1 & \lambda_2 & \lambda_3 & \cdots & \lambda_n \\ \lambda_1^2 & \lambda_2^2 & \lambda_3^2 & \cdots & \lambda_n^2 \\ \multicolumn{5}{c}{\dotfill} \\ \lambda_1^{n-1} & \lambda_2^{n-1} & \lambda_3^{n-1} & \cdots & \lambda_n^{n-1} \end{pmatrix},$$

Lemma. *If* $\{p_j(\lambda)\}_{j=0,\dots,N}$ *is a sequence of monic polynomials we have*

$$\det_{N\times N}(p_j(\lambda_k)) = \det_{N\times N}(\lambda_k^j) = \prod_{1\le j<k\le N}(\lambda_k - \lambda_j).$$

Proof.

- Let
$$p_j(\lambda) = \lambda^j + a_{j-1}\lambda^{j-1} + a_{j-2}\lambda^{j-2} + \cdots + a_0,$$
the coefficients a_ks may be different for each polynomial $p_j(\lambda)$.
- The determinant of a matrix does not change if we add to a row a linear combination of the other rows.
- $p_0(\lambda) = 1$ because the $p_j(\lambda)$s are monic. Thus the first line of (λ_k^j) is left untouched.
- If $p_{n-1}(\lambda) = \lambda^{n-1} + \sum_{j=0}^{n-2} a_j\lambda^j$ we multiply the first row by a_0, the second one by a_1, the third one by a_2 and so on. We then add them to the last row. The n-th row is now $(p_{n-1}(\lambda_1), p_{n-1}(\lambda_2), \ldots, p_{n-1}(\lambda_n))$
- We repeat the process for the $(n-1)$-th row and so on. $\square$

We are now ready to show

$$\boxed{\prod_{l=1}^{N} w(\lambda_l) \prod_{1\le j<k\le N} |\lambda_k-\lambda_j|^2 = \left(\prod_{l=0}^{N-1} (p_l,p_l)\right) \det_{N\times N}\left(K_N(\lambda_j,\lambda_k)\right),}$$

It is almost an immediate consequence of

$$\det_{N\times N}(p_j(\lambda_k)) = \det_{N\times N}(\lambda_k^j) = \prod_{1\le j<k\le N} (\lambda_k-\lambda_j).$$

and of the transposing lemma (see Conrey's lecture in 'Recent perspectives in RMT and Number Theory')

$$\det_{N\times N}(\psi_j(\lambda_k)) \det_{N\times N}(\phi_j(\lambda_k)) = \det_{N\times N}\left(\sum_{l=1}^{N} \psi_l(\lambda_j)\phi_l(\lambda_k)\right).$$

We have

$$\prod_{l=1}^{N} w(\lambda_l) \prod_{1\le j<k\le N} |\lambda_k-\lambda_j|^2 = \prod_{l=1}^{N} w(\lambda_l) \det_{N\times N}(p_j(\lambda_k)) \det_{N\times N}(p_j(\lambda_k))$$
$$= \prod_{l=0}^{N-1}(p_l,p_l) \det_{N\times N}\left(\left(\frac{w(\lambda_k)}{(p_j,p_j)}\right)^{1/2} p_j(\lambda_k)\right) \det_{N\times N}\left(\left(\frac{w(\lambda_k)}{(p_j,p_j)}\right)^{1/2} p_j(\lambda_k)\right).$$

By applying the transposing lemma, the r.h.s. of the above equation becomes

$$\prod_{l=0}^{N-1}(p_l,p_l) \det_{N\times N}\left(\big(w(\lambda_j)w(\lambda_k)\big)^{1/2} \sum_{l=0}^{N-1} \frac{p_l(\lambda_j)p_l(\lambda_k)}{(p_l,p_l)}\right)$$
$$= \prod_{l=0}^{N-1}(p_l,p_l) \det_{N\times N}\big(K_N(\lambda_j,\lambda_k)\big). \quad \square$$

We have now proved

$$\begin{aligned} P_{N2}(\lambda_1,\ldots,\lambda_N) &= C\prod_{j=1}^{N} w(\lambda_j) \prod_{1\le j<k\le N} \left|\lambda_k-\lambda_j\right|^2, \\ &= C\left(\prod_{l=0}^{N-1}(p_l,p_l)\right)\det_{N\times N}\left(K_N(\lambda_j,\lambda_k)\right) \end{aligned}$$

We are left to show

$$R_n(\lambda_1,\ldots,\lambda_n) = \det_{n\times n}\big(K_N(\lambda_j,\lambda_k)\big).$$

See also Conrey's lecture in 'Recent perspectives in RMT and Number Theory' for applications of these ideas to computing 1- and n-level densities in the classical compact groups.

Exercise:

1. Prove that

   $$K_N(\lambda,\mu) = \left(w(\lambda)w(\mu)\right)^{1/2} \sum_{j=0}^{N-1} \frac{p_j(\lambda)p_j(\mu)}{(p_j,p_j)}$$

   satisfies the hypotheses of Gaudin's lemma if and only if the polynomials $\{p_j(\lambda)\}$ are orthogonal.

2. Using Gaudin's lemma, compute the normalization constant C in

   $$P_{N2}(\lambda_1,\ldots,\lambda_N) = C \prod_{j=1}^{N} w(\lambda_j) \prod_{1\le j<k\le N} |\lambda_k - \lambda_j|^2 \,.$$

3. Prove that

   $$R_n(\lambda_1,\ldots,\lambda_n) = \det_{n\times n}\bigl(K_N(\lambda_j,\lambda_k)\bigr).$$

Gaudin's lemma. *Suppose that we have a function of two variables* $f(x,y)$ *which is integrable on an interval* J *and such that*

$$\int_J f(x,y)f(y,z)\mathrm{d}y = Kf(x,z),$$

for all x *and* z *and where* $K = K(f,J)$ *is a constant. Suppose also that*

$$\int_J f(y,y)\mathrm{d}y = D,$$

where $D = D(f,J)$ *is a constant as well. Then we have*

$$\int_J \det_{n\times n}(f(x_j,x_k))\mathrm{d}x_n = (D-(n-1)K)\det_{(n-1)\times(n-1)}(f(x_j,x_k))\,.$$

The GUE and the Hermite polynomials

- The GUE is the ensemble of matrices $H = (h_{jk})$ such that

  $$h_{jk} = \overline{h}_{kj}$$

  whose independent elements are i.i.d. normal random variables.
- As an example of the ideas just presented we will sketch the derivation of the Wigner's semi-circle law (1-level density).
- We will determine the kernel $K_N(\lambda, \mu)$ in the bulk of the spectrum as $N \to \infty$.
- We will show that it coincides with the kernel of Haar measure of $\mathrm{U}(N)$ as $N \to \infty$. (Example of universality.)

The semi-circle law

The j.p.d.f. for the eigenvalues of the GUE is

$$P_{N2}(\lambda_1,\ldots,\lambda_N) = C\prod_{j=1}^{N}\exp\left(-\lambda_j^2\right)\prod_{1\le j<k\le N}|\lambda_k-\lambda_j|^2\,.$$

The weighting factor is $w(\lambda)=\exp\left(-\lambda^2\right)$ the interval $J=\mathbb{R}$. The orthogonal polynomials are the ***Hermite polynomials.***

Their definition is

$$\boxed{H_n(\lambda) = (-1)^n\, e^{\lambda^2}\,\frac{d^n}{d\lambda^n}\,e^{-\lambda^2}.}$$

The Hermite polynomials satisfy the orthogonality relations

$$\boxed{\int_{-\infty}^{\infty} e^{-\lambda^2} H_m(\lambda) H_n(\lambda) d\lambda = \delta_{mn} 2^n n! \sqrt{\pi}}$$

The first few Hermite polynomials are

$$\begin{array}{lll} H_0(\lambda) = 1 & H_1(\lambda) = 2\lambda & H_2(\lambda) = 4\lambda^2 - 2 \\ H_3(\lambda) = 8\lambda^3 - 12\lambda & H_4(\lambda) = 16\lambda^4 - 48\lambda^2 + 12 & H_5(\lambda) = 32\lambda^5 - 160\lambda^3 + 120\lambda \end{array}$$

In general

$$H_n(\lambda) = n! \sum_{k=0}^{[n/2]} \frac{(-1)^k}{k!} \frac{(2\lambda)^{n-2k}}{(n-2k)!},$$

where [] denotes the integer part.

The normalization constant of the GUE is

$$C = \frac{1}{\left(\prod_{l=0}^{N-1}(p_l, p_l)\right) N!} = \frac{2^{-N(N-1)/1}}{\pi^{N/2}\prod_{j=1}^{N} k!}.$$

Therefore, the explicit form of the GUE Kernel is

$$K_N(\lambda, \mu) = \exp\Big(-(\lambda^2 + \mu^2)/2\Big)\frac{1}{\sqrt{\pi}}\sum_{j=0}^{N-1}\frac{H_j(\lambda)H_j(\mu)}{2^j j!},$$

(Remember that the $p_j(x)$ in Theorem 1 must be monic.)

The density of states is simply

$$\rho_N(\lambda) = \frac{e^{-\lambda^2}}{\sqrt{\pi}} \sum_{j=0}^{N-1} \frac{H_j(\lambda)^2}{2^j j!}$$

Using asymptotic properties of the Hermite polynomials one finds that

$$\rho_N(\lambda) \sim \begin{cases} \frac{\sqrt{2N}}{\pi}\sqrt{1-\lambda^2/(2N)} & \text{if } |\lambda| \le \sqrt{2N} \\ 0 & \text{if } |\lambda| > \sqrt{2N}. \end{cases}$$

This is known as Wigner's semi-circle law.

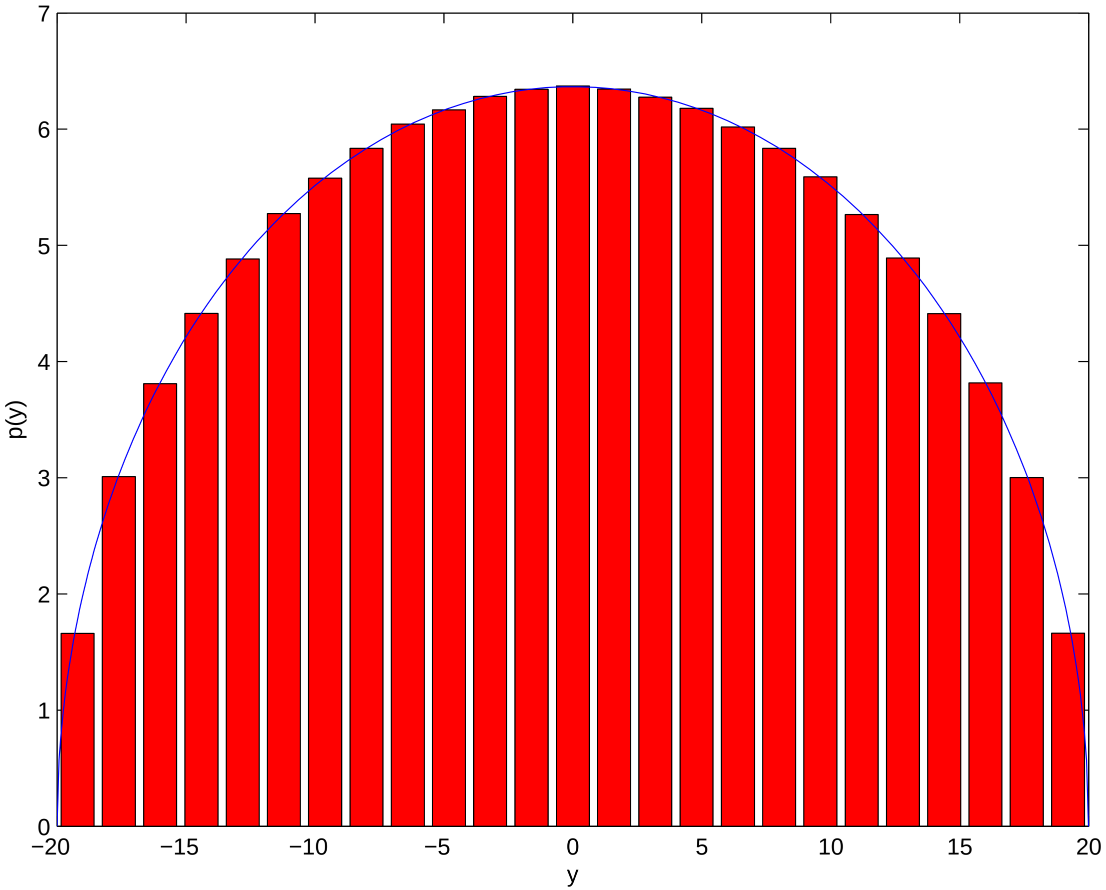

Figure 1: Numerical data obtained by averaging 5000 200 × 200 random matrices in the GUE compared with Wigner's semicircle law.

The ‘bulk’ correlation functions

As $N \to \infty$ we study the correlation functions in a neighbourhood of the origin small enough so that $\rho_N(\lambda)$ is approximately constant, but big enough to contain a large number of eigenvalues.

$$\rho_N(\lambda) = \frac{\sqrt{2N}}{\pi}\sqrt{1-\lambda^2/(2N)} = \frac{\sqrt{2N}}{\pi}\left(1-\frac{\lambda^2}{4N}+\cdots\right).$$

Therefore,

$$\rho_N(\lambda) \sim \frac{\sqrt{2N}}{\pi}, \quad N \to \infty.$$

Thus, we define

$$\boxed{x = \rho(0)\lambda = \frac{\sqrt{2N}}{\pi}\lambda.}$$

The average number of the rescaled eigenvalues in unit interval is 1.

We have

$$R_n^{\mathrm{GUE}}(\lambda_1, \ldots, \lambda_N) = \det_{n\times n}\big(K_N(\lambda_j, \lambda_k)\big).$$

Therefore, we need to compute

$$\begin{aligned}\lim_{N\to\infty}\left(\frac{\pi}{\sqrt{2N}}\right)^n \det_{n\times n}\left(K_N\left(\frac{\pi x_j}{\sqrt{2N}}, \frac{\pi x_k}{\sqrt{2N}}\right)\right)&\\ = \lim_{N\to\infty}\det_{n\times n}\left(\frac{\pi}{\sqrt{2N}}K_N\left(\frac{\pi x_j}{\sqrt{2N}}, \frac{\pi x_k}{\sqrt{2N}}\right)\right)&\end{aligned}$$

We need to evaluate the asymptotics as $N \to \infty$ of the kernel

$$\boxed{\begin{aligned}\frac{\pi}{\sqrt{2N}}K_N\left(\frac{\pi x}{\sqrt{2N}}, \frac{\pi y}{\sqrt{2N}}\right) &= \sqrt{\frac{\pi}{2N}}\exp\left(-\frac{\pi^2\left(x^2+y^2\right)}{4N}\right)\\ &\times \sum_{j=0}^{N-1}\frac{H_j\left(\pi x/\sqrt{2N}\right)H_j\left(\pi y/\sqrt{2N}\right)}{2^j j!}\end{aligned}}$$

The Christoffel-Darboux formula

The essential tool in the theory of orthogonal polynomials is the ***Christoffel-Darboux formula:***

Theorem. *If $\{p_k(x)\}$ is a system of orthogonal polynomials (not necessarily monic) the following identity holds:*

$$\sum_{j=0}^{N-1} \frac{p_j(\lambda)p_j(\mu)}{(p_j, p_j)} = \frac{1}{A_N(p_{N-1}, p_{N-1})} \frac{p_N(\lambda)p_{N-1}(\mu) - p_{N-1}(\lambda)p_N(\mu)}{\lambda - \mu},$$

where $A_n = a_n/a_{n-1}$ and a_j is the coefficient of λ^j in $p_j(\lambda)$.

Applying the Christoffel-Darboux formula to Hermite polynomials gives

$$\sum_{j=0}^{N-1} \frac{H_j(\lambda)H_j(\mu)}{2^j\Gamma(j+1)} = \frac{1}{2^N\Gamma(N)} \frac{H_N(\lambda)H_{N-1}(\mu) - H_{N-1}(\lambda)H_N(\mu)}{\lambda-\mu}, \tag{2}$$

where

$$\Gamma(z) = \int_0^\infty e^{-t}t^{z-1}dt,$$

which satisfies the identities

$$\Gamma(z+1) = z\Gamma(z) \quad \text{and} \quad \Gamma(m+1) = m!, \quad m \in \mathbb{N}.$$

We are interest in the limit $N \to \infty$ of the kernel.

- Consider first

$$e^{-\left(\lambda^2+\mu^2\right)/2}\,\frac{H_N(\lambda)H_{N-1}(\mu)}{2^N\Gamma(N)}. \tag{3}$$

Using the asymptotic formula

$$\frac{\Gamma\left(N/2+1\right)}{\Gamma\left(N+1\right)}\,e^{-\lambda^2/2}H_N(\lambda) = \cos\left(\sqrt{2N+1}\,\lambda - N\pi/2\right) + \mathrm{O}\left(N^{-1/2}\right),$$

Eq. (3) becomes

$$\boxed{-\frac{\Gamma(N+1)\cos\left(\sqrt{2N+1}\,\lambda - \frac{N\pi}{2}\right)\sin\left(\sqrt{2N-1}\,\mu - \frac{N\pi}{2}\right)}{2^N\Gamma\left(\frac{N}{2}+1\right)\Gamma\left(\frac{N}{2}+\frac{1}{2}\right)} + \mathrm{O}\left(N^{-1/2}\right),}$$

where we have used $\cos(\theta+\pi/2) = -\sin(\theta)$.

- Note that

$$\cos\left(\sqrt{2N+1}\,\lambda - N\pi/2\right) = \cos\left(\sqrt{2N}\,\lambda - N\pi/2\right) + \mathrm{O}\left(N^{-1/2}\right),$$

$$\sin\left(\sqrt{2N+1}\,\mu - N\pi/2\right) = \sin\left(\sqrt{2N}\,\mu - N\pi/2\right) + \mathrm{O}\left(N^{-1/2}\right).$$

Thus, we can write

$$e^{-\left(\lambda^2+\mu^2\right)/2}\,\frac{H_N(\lambda)H_{N-1}(\mu)}{2^N\Gamma(N)} \sim -\frac{\Gamma\left(N+1\right)}{2^N\Gamma\left(\frac{N}{2}+1\right)\Gamma\left(\frac{N}{2}+\frac{1}{2}\right)} \times \cos\left(\sqrt{2N}\,\lambda - N\pi/2\right)\sin\left(\sqrt{2N}\,\mu - N\pi/2\right) \tag{4}$$

as $N \to \infty$.

It turns out that

$$\boxed{\frac{\Gamma\left(N+1\right)}{2^{N}\,\Gamma\left(\frac{N}{2}+1\right)\Gamma\left(\frac{N}{2}+\frac{1}{2}\right)}=\frac{1}{\sqrt{\pi}}.}$$

This identity is the duplication formula for $\Gamma(z)$ rearranged:

$$\Gamma(2z)=(2\pi)^{-\frac{1}{2}}2^{2z-\frac{1}{2}}\Gamma(z)\Gamma\big(z+\tfrac{1}{2}\big). \tag{5}$$

Indeed, by setting $N=2z$ we obtain

$$\frac{\Gamma\left(2z+1\right)}{2^{2z}\,\Gamma\left(z+1\right)\Gamma\left(z+\frac{1}{2}\right)}=\frac{\Gamma\left(2z\right)}{2^{2z-1}\Gamma\left(z\right)\Gamma\left(z+\frac{1}{2}\right)}=\frac{1}{\sqrt{\pi}}.$$

Combining the formula in the red box with Eq. (4) we have

$$
\begin{aligned}
K_N(\lambda,\mu) &= \frac{1}{\sqrt{\pi}}\, e^{-\left(\lambda^2+\mu^2\right)/2} \sum_{j=0}^{N-1} \frac{H_j(\lambda)H_j(\mu)}{2^j\Gamma(j+1)} \\
&\sim \frac{1}{\pi}\frac{\sin\left(\sqrt{2N}\,\lambda - N\pi/2\right)\cos\left(\sqrt{2N}\,\mu - N\pi/2\right)}{\lambda-\mu} \\
&\quad - \frac{1}{\pi}\frac{\sin\left(\sqrt{2N}\,\mu - N\pi/2\right)\cos\left(\sqrt{2N}\,\lambda - N\pi/2\right)}{\lambda-\mu} \\
&\sim \frac{1}{\pi}\frac{\sin\left(\sqrt{2N}(\lambda-\mu)\right)}{\lambda-\mu}, \quad N\to\infty
\end{aligned}
$$

By setting

$$\rho_N(0)\,(\lambda-\mu) = \frac{\sqrt{2N}}{\pi}\,(\lambda-\mu) = r$$

we obtain the bulk kernel

$$S(r) = \lim_{N\to\infty} \frac{\pi}{\sqrt{2N}}\, K_N(\lambda,\mu) = \frac{\sin(\pi r)}{\pi r}.$$

Therefore, the GUE bulk correlation functions are

$$\begin{aligned} R_n(x_1,\ldots,x_n) &= \lim_{N\to\infty} \left(\frac{\pi}{\sqrt{2N}}\right)^n R_n^{\mathrm{GUE}}\left(\frac{\pi x_1}{\sqrt{2N}},\ldots,\frac{\pi x_n}{\sqrt{2N}}\right) \\ &= \det_{n\times n}\big(S(r_{kj})\big), \quad r_{kj} = x_k - x_j. \end{aligned}$$

A remark on universality:

We have just proved that the GUE bulk correlation functions are the same as the CUE ones in the limit $N \to \infty$. In random matrix theory it has long been conjectured that in the limit $N \to \infty$ the local correlations of the eigenvalues of random matrices depend exclusively on the invariance properties of the probability distribution that defines the ensemble and not on the explicit form of the measure itself. Mathematically it translates into the statement that the local correlations are mainly determined by the absolute value of (powers) of the Vandermonde in the j.p.d.f. for the eigenvalues, whose origin is essentially geometrical. Provided that the local eigenvalue density has the same asymptotic behaviour.

- The CUE measure is invariant under the map $U \mapsto VUW$.
- The GUE measure is invariant under the map $H \mapsto VHV^*$.