Asymptotics of averages over the classical compact groups

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- Let $\mathrm{G}(N)$ denote one of the classical compact groups $\mathrm{U}(N)$, $\mathrm{O}(N)$ and $\mathrm{Sp}(2N)$ endowed with Haar measure $d\mu_{\mathrm{G}(N)}$.
- Let also $f : \mathrm{G}(N) \longrightarrow \mathbb{C}$.

We will be concerned with the asymptotics as $N \to \infty$ of the integrals

$$\boxed{\mathbb{E}_{\mathrm{G}(N)}[f] = \langle f \rangle_{\mathrm{G}(N)} = \int_{\mathrm{G}(N)} f(U) d\mu(U)}$$

Why are these integrals important? $\zeta(\frac{1}{2}+it)$ for large t can be modelled by characteristic polynomials of random unitary matrices.

- Averages over $t \Longleftrightarrow$ averages over $\mathrm{U}(N)$.
- Averages over family of L-functions correspond to averages over the classical compact groups.

Example:

- Moments of characteristic polynomials:

$$\int_{\mathrm{G}(N)} |Z(U,z)|^{2k}\, d\mu(U)$$

Used to study moments and value distributions of L-functions.

Example:

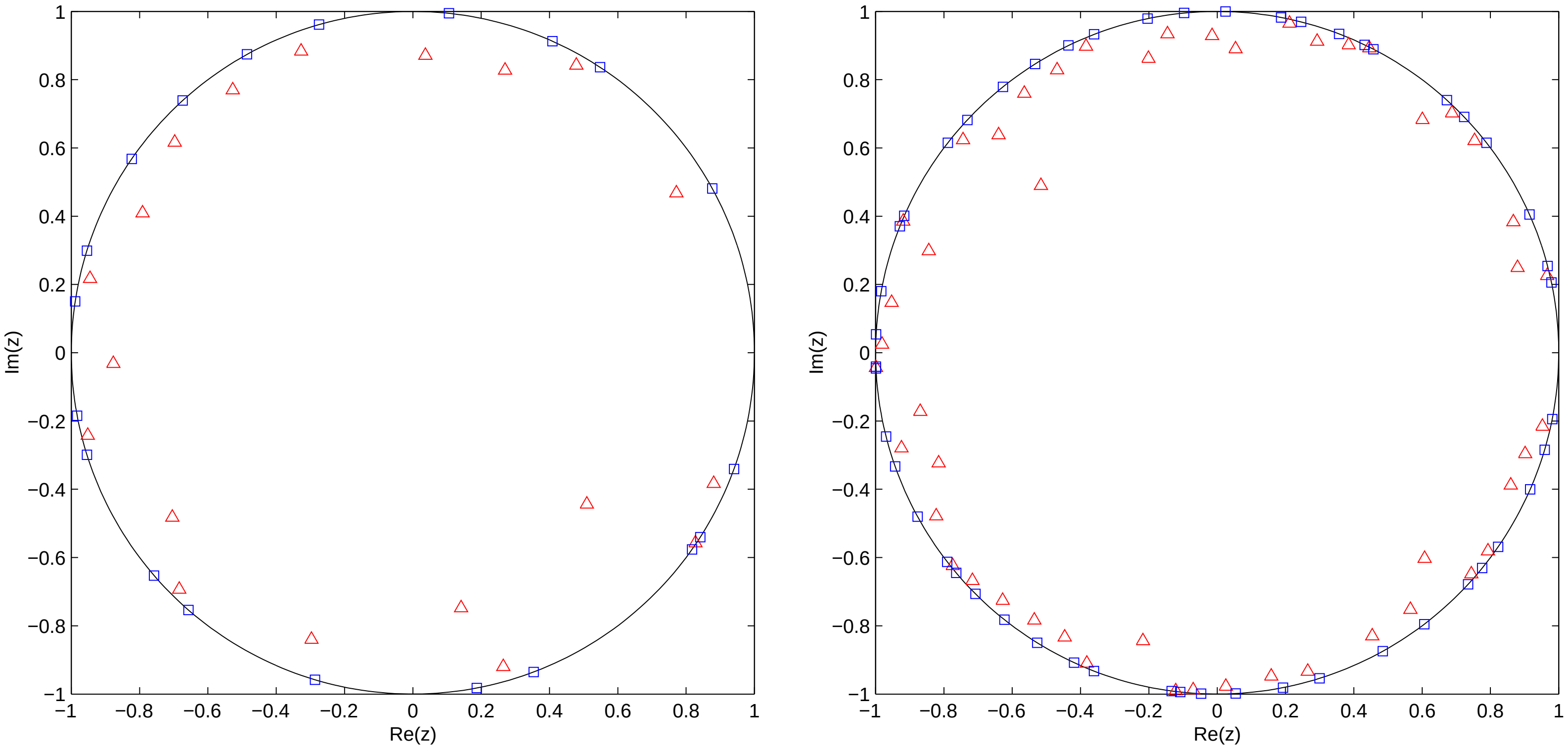
- The density $\rho(z)$ of the roots of the derivative of characteristic polynomials of random unitary matrices:

$$\rho(z) = \frac{1}{4\pi^2(N-1)} \int_{\mathbb{C}} \left(\int_{\mathrm{U}(N)} f(U,z,w) d\mu(U) \right) d^2 w,$$

where

$$f(U,z,w) = \exp\left[\frac{i}{2} \left(\frac{Z'(U,z)}{Z(U,z)} \overline{w} + \frac{\overline{Z'(U,z)}}{\overline{Z(U,z)}} w \right) \right]$$
$$\times \left| \frac{d}{dz} \left[Z'(U,z)/Z(U,z) \right] \right|^2$$

$\rho(z)$ is believed to model the distribution of the zeros of $\zeta'(s)$ to the right of the critical line.



(a) $N = 20$ (b) $N = 50$

Figure 1: Zeros of characteristic polynomials of random unitary matrices (□) and of their derivatives (△).

Example:

- $e^{i\theta_j}$, $1 \leq j \leq N$, be the eigenvalues of $U \in \mathrm{U}(N)$ and let $g(\theta)$ be a 2π-periodic test function. What is the value distribution of the linear statistic

$$\mathrm{Tr}\, g(U) := \sum_{j=1}^{N} g(\theta_j)?$$

We need to study the *cumulant* expansion of $\mathrm{Tr}\, g(U)$

$$\log \mathbb{E}_{\mathrm{U}(N)} \left[\exp\left(\lambda\, \mathrm{Tr}\, g(U)\right)\right] = \sum_{j=1}^{\infty} C_j^{\mathrm{U}(N)} \frac{\lambda^j}{j!}.$$

Corresponding question in number theory:

what is the distribution of the number of zeros of the Riemann zeta function lying in an interval of size h around hight T?

Averages over $\mathrm{U}(N)$

Let $f : \mathrm{U}(N) \longrightarrow \mathbb{C}$ be a class function, i.e.

$$f(V^*UV) = f(U) \quad V \in \mathrm{U}(N). \tag{1}$$

In other words

$$f(U) = f(\theta_1, \ldots, \theta_N).$$

Exercise:

Prove that implies that Eq. (1) implies

$$f(\theta_1, \ldots, \theta_N) = f(\theta_{\sigma 1}, \ldots, \theta_{\sigma N}), \quad \sigma \in S_N$$

We now want to learn how to compute integrals of the type

$$\boxed{\mathbb{E}_{\mathrm{U}(N)}[f] = \int_{\mathrm{U}(N)} f(U) d\mu(U).}$$

Using Weyl's integration formula, we obtain

$$\mathbb{E}_{\mathrm{U}(N)}[f] = \frac{1}{(2\pi)^N N!} \int_{[0,2\pi)^N} f(\theta_1, \dots, \theta_N) \prod_{1 \le j < k \le N} \left| e^{i\theta_k} - e^{i\theta_j} \right|^2 d\theta_1 \cdots d\theta_N$$

Let g be 2π-periodic. We shall consider function of the type

$$f(\theta_1, \dots, \theta_N) = \prod_{j=1}^{N} g(\theta_j).$$

The starting point is ***Heine's identity:***

$$\frac{1}{(2\pi)^N N!}\int_{[0,2\pi)^N}\left(\prod_{j=1}^{N} g(\theta_j)\right)\prod_{1\le j<k\le N}\left|e^{i\theta_k}-e^{i\theta_j}\right|^2 d\theta_1\cdots d\theta_N$$
$$=\det_{N\times N}\left(\frac{1}{2\pi}\int_0^{2\pi} g(\theta)e^{i(j-k)\theta}d\theta\right) \qquad (2)$$

Let $\hat{g}_l = \frac{1}{2\pi}\int_0^{2\pi} g(\theta)e^{-il\theta}d\theta.$

$T_N[g] = (\hat{g}_{j-k})$ is called *Toeplitz matrix; $g(\theta)$ its symbol.*

Its determinant $D_N[g] = \det_{N\times N}(\hat{g}_{j-k})$ is called *Toeplitz determinant.*

Explicitly

$$T_N[g] = \begin{pmatrix} \hat{g}_0 & \hat{g}_{-1} & \hat{g}_{-2} & \hat{g}_{-3} & \cdots & \hat{g}_{1-N} \\ \hat{g}_1 & \hat{g}_0 & \hat{g}_{-1} & \hat{g}_{-2} & \cdots & \hat{g}_{2-N} \\ \hat{g}_2 & \hat{g}_1 & \hat{g}_0 & \hat{g}_{-1} & \cdots & \hat{g}_{3-N} \\ \vdots & \ddots & \ddots & \ddots & \ddots & \\ \hat{g}_{N-2} & & \ddots & \ddots & \ddots & \hat{g}_{-1} \\ \hat{g}_{N-1} & \cdots & \cdots & \hat{g}_2 & \hat{g}_1 & \hat{g}_0 \end{pmatrix}.$$

Concisely, we can write

$$\boxed{\mathbb{E}_{\mathrm{U}(N)}\left[\prod_{j=1}^{N} g(\theta_j)\right] = D_N[g]}$$

Heine's identity is a particular case of

Andréief's identity. *For any interval J and integrable functions $\psi_j(x)$ and $\phi_j(x)$, $1 \le j \le N$, we have*

$$\frac{1}{N!}\int_{J^n} \det_{N\times N}(\psi_j(x_k)) \det_{N\times N}(\phi_j(x_k))dx_1\cdots dx_N$$

$$= \det_{N\times N}\left(\int_J \psi_j(x)\phi_k(x)dx\right)$$

(For a proof see Conrey's article in 'Recent Perspectives in Random Matrix Theory and Number Theory'.)

Exercise

Prove Heine's identity.

Use Andréief's identity and the fact the determinant of the Vandermonde matrix

$$V = \begin{pmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^{N-1} \\ 1 & x_2 & x_2^2 & \cdots & x_2^{N-1} \\ 1 & x_3 & x_3^2 & \cdots & x_3^{N-1} \\ \hdotsfor{5} \\ 1 & x_N & x_N^2 & \cdots & x_N^{N-1} \end{pmatrix} \tag{3}$$

is given by

$$\det V = \Delta_N(x_1, \ldots, x_N) = \prod_{1 \le j < k \le N} (x_k - x_j)$$

Exercise

Let $U \in \mathrm{U}(N)$, f and h be class functions defined by

$$f(U) = \prod_{j=1}^{N} (1 + \cos\theta_j) \quad \text{and} \quad h(U) = \prod_{j=1}^{N} \cos\theta_j$$

Prove that

$$\mathbb{E}_{\mathrm{U}(N)}[f] = \frac{1+N}{2^N}$$

and

$$\mathbb{E}_{\mathrm{U}(N)}[h] = \begin{cases} 0 & \text{if } N \text{ is odd} \\ \left(-\frac{1}{4}\right)^m & \text{if } N = 2m \text{ with } m \geq 1. \end{cases}$$

The strong Szegő limit theorem. *Let*

$$g(\theta) = \sum_{k=-\infty}^{\infty} \hat{g}_k \, \mathrm{e}^{\mathrm{i}k\,\theta}$$

be a complex smooth function on the unit circle. If the series

$$\sum_{k=-\infty}^{\infty} |\hat{g}_k| \quad \textit{and} \quad \sum_{k=-\infty}^{\infty} |k|\,|\hat{g}_k|^{2}$$

converge, then

$$\boxed{\log D_N[\exp(g)] = \hat{g}_0 N + \sum_{k=1}^{\infty} k\,\hat{g}_{-k}\hat{g}_k + \mathrm{o}(1), \quad N \to \infty.}$$

How is Szegő's theorem related to averages over U(N)?

Heine's identity tells us that

$$\mathbb{E}_{\mathrm{U}(N)}\left[\prod_{j=1}^{N}\exp\big(\lambda g(\theta_j)\big)\right] = \mathbb{E}_{\mathrm{U}(N)}\left[\exp\left(\lambda\sum_{j=1}^{N} g(\theta_j)\right)\right]$$

$$= \mathbb{E}_{\mathrm{U}(N)}\Big[\exp\big(\lambda\,\mathrm{Tr}\,g(U)\big)\Big] = D_N\big[\exp(\lambda g)\big].$$

This is the moment generating function of the linear statistic

$$\mathrm{Tr}\,g(U) = \sum_{j=1}^{N} g(\theta_j).$$

The *cumulant-generating function* of $\operatorname{Tr} g(U)$ is

$$\log \mathbb{E}_{\mathrm{U}(N)}\left[\exp\left(\lambda \operatorname{Tr} g(U)\right)\right] = \sum_{j=1}^{\infty} C_j^{\mathrm{U}(N)} \frac{\lambda^j}{j!}.$$

Szegő's theorem is equivalent to

$$\begin{aligned}\log \mathbb{E}_{\mathrm{U}(N)}\left[\exp\left(\lambda \operatorname{Tr} g(U)\right)\right] &= \log D_N\left[\exp(\lambda g)\right] \\ &= \lambda \hat{g}_0 N + \lambda^2 \sum_{k=1}^{\infty} k\, \hat{g}_{-k}\hat{g}_k + \mathrm{o}(1), \quad N \to \infty.\end{aligned}$$

Szegő's theorem gives us the leading order asymptotics of the first two cumulants:

$$C_1^{\mathrm{U}(N)} = N\hat{g}_0, \quad C_2^{\mathrm{U}(N)} \sim 2\sum_{k=1}^{\infty} k\, \hat{g}_{-k}\hat{g}_k, \quad N \to \infty.$$

We now from probability theory that:

- the first cumulant is the *mean* of a probability distribution;
- the second cumulant is the *variance* of a probability distribution.

Let $e^{i\theta_j}$ be the eigenvalues of a random unitary matrix U with distribution given by Haar measure and let $g(\theta)$ be a 2π-periodic function. Szegő's theorem gives us the mean and variance as $N \to \infty$ of the random variable

$$\operatorname{Tr} g(U) = \sum_{j=1}^{N} g(\theta_j).$$

Exercise:

Compute the leading order term as $N \to \infty$ of the integral

$$\int_{\mathrm{U}(N)} \exp\bigl(\operatorname{Tr} g(U)\bigr) d\mu(U),$$

where

$$g(\theta) = 2\frac{1 - \alpha\cos(\theta)}{1 + \alpha^2 - 2\alpha\cos(\theta)}, \quad \alpha \in \mathbb{R}, \quad |\alpha| < 1.$$

The most simple linear statistics of the eigenvalues $e^{i\theta_j}$, $1 \leq j \leq N$, is the trace of U:

$$\operatorname{Tr} U = e^{i\theta_1} + \cdots + e^{i\theta_N}.$$

Fix an integer m and consider the random variables $\operatorname{Re} \operatorname{Tr} U^k$ and $\operatorname{Im} \operatorname{Tr} U^k$, $k = 1, \ldots, m$.

Corollary. *If* $U \in \mathrm{U}(N)$ *with probability distribution given by Haar measure, then*

$$\sqrt{\frac{2}{k}} \operatorname{Re} \operatorname{Tr} U^k \quad \textit{and} \quad \sqrt{\frac{2}{k}} \operatorname{Im} \operatorname{Tr} U^k,$$

for $k = 1, \ldots, m$, *converge in distribution to independent standard normal random variables as* $N \to \infty$.

Sketch of the proof. The moment generating function $M_X(\lambda)$ of a random variable X (if it exists) is defined by

$$M_X(\lambda) = \mathbb{E}\left[e^{\lambda X}\right].$$

In the case of a real random variable X with probability density function $p(x)$, it is simply

$$M_X(\lambda) = \int_{-\infty}^{\infty} p(x) e^{\lambda x} dx.$$

If $Y = \sum_{k=1}^{m} c_k X_k$ *is the sum of independent random variable, then*

$$M_Y(\lambda) = M_{X_1}(\lambda c_1) M_{X_2}(\lambda c_2) \cdots M_{X_m}(\lambda c_m)$$

and viceversa.

Fixing an integer m take the trigonometric polynomial

$$g(\theta) = \sum_{k=1}^{m} \xi_k \cos(k\theta) + \chi_k \sin(k\theta).$$

Then consider the following sum of random variables:

$$\begin{aligned} \operatorname{Tr} g(U) &= \sum_{j=1}^{N} g(\theta_j) = \sum_{j=1}^{N} \sum_{k=-\infty}^{\infty} \hat{g}_k e^{ik\theta_j} = \sum_{k=-\infty}^{\infty} \hat{g}_k \operatorname{Tr} U^k \\ &= \sum_{k=1}^{m} \xi_k \operatorname{Re} \operatorname{Tr} U^k + \chi_k \operatorname{Im} \operatorname{Tr} U^k. \end{aligned}$$

The moment generating function of $\operatorname{Tr} g(U)$ is

$$\boxed{\mathbb{E}_{\mathrm{U}(N)} \Big[\exp\big(\lambda \operatorname{Tr} g(U)\big) \Big] = \int_{\mathrm{U}(N)} \exp\big(\lambda \operatorname{Tr} g(U)\big) d\mu(U)}$$

We then apply Szegő's theorem.

$\hat{g}_0 = 0$, therefore the mean of $\operatorname{Tr} g(U)$ is zero.

Its variance instead is given by

$$\sigma^2 = 2\sum_{k=1}^{\infty} k\,\hat{g}_{-k}\hat{g}_k = \sum_{k=1}^{m} \left(k\xi_k^2 + k\chi_k^2\right)/2$$

Taking the limit as $N \to \infty$, we have

$$\boxed{\begin{aligned}\lim_{N\to\infty} \int_{\mathrm{U}(N)} \exp\big(\lambda \operatorname{Tr} g(U)\big) d\mu(U) &= \exp\left(\lambda^2 \sum_{k=1}^{m} \left(k\xi_k^2 + k\chi_k^2\right)/4\right) \\ &= \prod_{k=1}^{m} \exp\left(k\lambda^2\xi_k^2/4\right) \exp\left(k\lambda^2\chi_k^2/4\right).\end{aligned}}$$

The moment generating function of a normal random variable X with mean $\mu = 0$ and variance σ^2 is

$$M_X(\lambda) = \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\infty} e^{-\frac{x^2}{2\sigma^2} + \lambda x} dx = e^{\frac{\sigma^2 \lambda^2}{2}}.$$

Therefore, as $N \to \infty$ *the moment generating function of*

$$\sum_{k=1}^{m} \xi_k \operatorname{Re} \operatorname{Tr} U^k + \chi_k \operatorname{Im} \operatorname{Tr} U^k.$$

factorizes into the product of the generating functions of normal random variables with $\mu = 0$ *and* $\sigma^2 = k/2$*.*

The other compact groups

- Let us start with $\mathrm{SO}^+(2N)$
- Eigenvalues come in complex conjugate pairs:

$$e^{i\theta_1}, e^{-i\theta_1}, \ldots, e^{i\theta_N}, e^{-i\theta_N}$$

- Haar measure expressed in terms of the eigenvalues is

$$P(\theta_1, \ldots, \theta_N) = \frac{1}{Z} \prod_{1 \le j < k \le N} (\cos\theta_k - \cos\theta_j)^2$$

- We want to look at integrals of the form

$$\boxed{\frac{1}{Z}\int_{-\pi}^{\pi} \cdots \int_{-\pi}^{\pi} \prod_{j=1}^{N} f(\theta_j) f(-\theta_j) \prod_{1 \le j < k \le N} (\cos\theta_k - \cos\theta_j)^2 d\theta_1 \cdots d\theta_N}$$

We have the analogue of Heine's identity for $\mathrm{U}(N)$ (prove it!):

$$\frac{1}{Z}\int_{-\pi}^{\pi}\cdots\int_{-\pi}^{\pi}\prod_{j=1}^{N} f(\theta_j)f(-\theta_j)$$

$$\times \prod_{1\le j<k\le N}(\cos\theta_k-\cos\theta_j)^2 d\theta_1\cdots d\theta_N = \det_{N\times N}(\alpha_{jk})$$

If we set $g(\theta)=f(\theta)f(-\theta)$, we have

$$\alpha_{00} = \frac{1}{2\pi}\int_0^{2\pi} g(\theta)d\theta = g_0,$$

$$\alpha_{0j} = \alpha_{j0} = \frac{\sqrt{2}}{\pi}\int_0^{\pi} g(\theta)\cos(j\,\theta)d\theta = \sqrt{2}g_j, \quad j>0,$$

$$\alpha_{jk} = \frac{2}{\pi}\int_0^{\pi} g(\theta)\cos(j\,\theta)\cos(k\,\theta)d\theta = g_{j-k}+g_{j+k}, \quad j,k>0$$

where $g_j = \frac{1}{2\pi}\int_0^{2\pi} g(\theta)\mathrm{e}^{-ij\,\theta}d\theta$.

The expectation value

$$\mathbb{E}_{\mathrm{SO}^+(2N)}\left[\prod_{j=1}^{N} f(\theta)f(-\theta)\right] = \det_{N\times N}(\alpha_{jk}),$$

where α_{jk} is the sum of a Toeplitz and a Henkel matrix.

Let us rewrite

$$\frac{1}{Z}\int_{-\pi}^{\pi}\cdots\int_{-\pi}^{\pi}\prod_{j=1}^{N} f(\theta_j)f(-\theta_j)\prod_{1\le j<k\le N}(\cos\theta_k-\cos\theta_j)^2 d\theta_1\cdots d\theta_N$$

$$=\frac{1}{Z}\int_{-\pi}^{\pi}\cdots\int_{-\pi}^{\pi}\left(\prod_{l=1}^{N} g(\theta_l)\right)\det_{N\times N}\left(\cos^j\theta_k\right)^2 d\theta_1\cdots d\theta_N$$

Let us set $x_j = \cos\theta_j$. Our average becomes

$$\frac{1}{Z}\int_{-1}^{1}\cdots\int_{-1}^{1}\left(\prod_{l=1}^{N} g(\cos^{-1} x_l)(1-x_l^2)^{-1/2}\right)\det_{N\times N}\left(x_j^k\right)^2 dx_1\cdots dx_N$$

$$= \frac{1}{Z}\int_{-1}^{1}\cdots\int_{-1}^{1}\left(\prod_{l=1}^{N} g(\cos^{-1} x_l)(1-x_l^2)^{-1/2}\right)$$

$$\times \det_{N\times N}\left(p_k^{(-1/2,-1/2)}(x_j)\right)^2 dx_1\cdots dx_N$$

The $p^{(-1/2,-1/2)}(x)$ are orthogonal polynomials with respect to the weight $w(x) = (1-x^2)^{-1/2}$:

$$\int_{-1}^{1}(1-x^2)^{-1/2} p_j^{(-1/2,-1/2)}(x) p_k^{(-1/2,-1/2)}(x)dx = \delta_{jk}.$$

The polynomials $p_j^{(-1/2,-1/2)}(x)$ are the Chebyshev polynomials.

$$\frac{1}{Z}\int_{-\pi}^{\pi}\cdots\int_{-\pi}^{\pi}\left(\prod_{l=1}^{N} g(\theta_l)\right)\det_{N\times N}\left(\cos^{j}\theta_k\right)^2 d\theta_1\cdots d\theta_N$$

$$=\frac{1}{Z}\int_{-1}^{1}\cdots\int_{-1}^{1}\left(\prod_{l=1}^{N} g(\cos^{-1}x_l)(1-x_l^2)^{-1/2}\right)$$

$$\times\det_{N\times N}\left(p_k^{(-1/2,-1/2)}(x_j)\right)^2 dx_1\cdots dx_N=\det_{N\times N}(\alpha_{jk}),$$

where the α_{jk} are the same as before and

$$\boxed{\alpha_{jk}=\int_{-1}^{1} g(\cos^{-1}x)(1-x^2)^{-1/2}p_j^{(-1/2,-1/2)}(x)p_k^{(-1/2,-1/2)}(x)dx.}$$

In general we want to compute the average

$$\mathbb{E}_{\mathrm{G}(\tilde{N})}\left[\prod_{j=1}^{\tilde{N}} f(\theta_j)\right],$$

where $\mathrm{G}(\tilde{N})$ refers to any of $\mathrm{SO}^+(2N)$, $\mathrm{SO}^-(2N+2)$, $\mathrm{Sp}(2N)$, $\mathrm{SO}^{\pm}(2N+1)$ and $\tilde{N}$ is the total number of eigenvalues.

Haar measure is

$$\boxed{\begin{aligned} P^N_{(\sigma_1,\sigma_2)}(\theta_1,\ldots,\theta_N) = \frac{1}{Z_N^{(\sigma_1,\sigma_2)}} \prod_{l=1}^{N} (1+\cos\theta_l)^{\sigma_1+1/2}(1-\cos\theta_l)^{\sigma_2+1/2} \\ \times \prod_{1\le j<k\le N} (\cos\theta_j - \cos\theta_k)^2, \end{aligned}}$$

by setting $x_j = \cos\theta_j$ we have

$$\frac{1}{Z^{(\sigma_1,\sigma_2)}} \int_{-1}^{1} \cdots \int_{-1}^{1} \left(\prod_{j=1}^{N} g(\cos^{-1} x_j) \right) \prod_{j=1}^{N} (1+x_j)^{\sigma_1} (1-x_j)^{\sigma_2} \times \prod_{1\le j<k\le N} (x_j - x_k)^2 \, dx_1 \cdots dx_N = \det_{N\times N} (\alpha_{jk}^{(\sigma_1,\sigma_2)}), \quad (9)$$

$$\alpha_{jk}^{(\sigma_1,\sigma_2)} = \int_{-1}^{1} g(\cos^{-1} x) w_{(\sigma_1,\sigma_2)}(x) p_j^{(\sigma_1,\sigma_2)}(x) p_k^{(\sigma_1,\sigma_2)}(x) dx.$$

The $\{p_j^{(\sigma_1,\sigma_2)}(x)\}$ are called *Jacobi polynomials* and are orthogonal with respect to the weight

$$\boxed{w_{(\sigma_1,\sigma_2)}(x) = (1+x_j)^{\sigma_1} (1-x_j)^{\sigma_2} .}$$