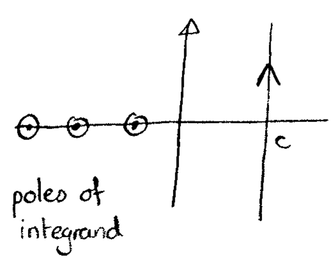
Keating 3

Let $A \in U(N)$

$$Z(A,\theta) = \det(I - Ae^{-i\theta})$$

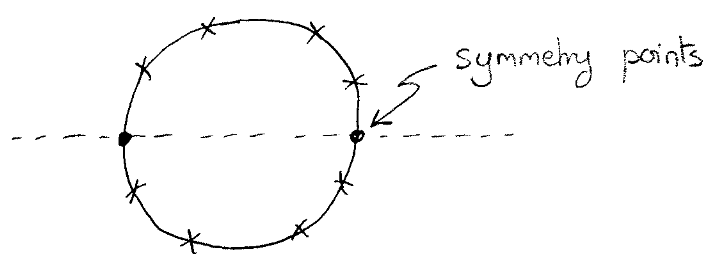
$$\mathbb{E}_{U(N)}\left(\delta(|Z(A,\theta)| - w)\right) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \prod_{j=1}^{N} \frac{\Gamma(j+t)\Gamma(j+t)}{\Gamma(j+2t)^2}\frac{1}{w^{t+1}}\,dt$$



$$\mathbb{E}_{U(N)}\left(\delta(|Z(A,\theta)| - w)\right) \longrightarrow \text{const} \quad \text{as} \quad w \to 0.$$

For $|\zeta(\frac{1}{2}+it)|$ take $N = \log\frac{t}{2\pi}$, and include the arithmetical factor in the moments.

Let $A \in SO(2N)$. Eigenvalues come in complex conjugate pairs



$$Z(A,\theta) = \prod_{n=1}^{N}(1-e^{i(\theta_n-\theta)})(1-e^{-i\theta_n-i\theta})$$

$$\mathbb{E}_{SO(2N)} |Z(A,0)|^s = 2^{Ns} \prod_{j=1}^{N} \frac{\Gamma(N+j-1)\Gamma(s+j-\frac{1}{2})}{\Gamma(j-\frac{1}{2})\Gamma(s+j+N-1)}$$

$$=: M_0(s,N)$$

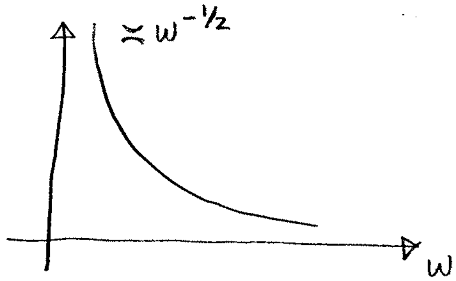
From this it follows that

- $\log|Z|$ satisfies a central limit theorem as $N \to \infty$

$$\mathbb{E}_{SO(2N)} \left(\delta(|Z(A,0)| - w) \right) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} M_0(s,N) \frac{1}{W^{s+1}} \, ds$$

As $w \to 0$ this is $\sim h(N) w^{-1/2}$.

A calculation shows that $h(N) \sim 2^{-7/8} G(\frac{1}{2}) \pi^{-1/4} N^{3/8}$ as $N \to \infty$



Application to ranks of elliptic curves (Conrey, Keating, Rubinstein, Snaith) 2002, 2006

$E: \; y^2 = x^3 + ax + b$. $\qquad r(E)$ - rank.

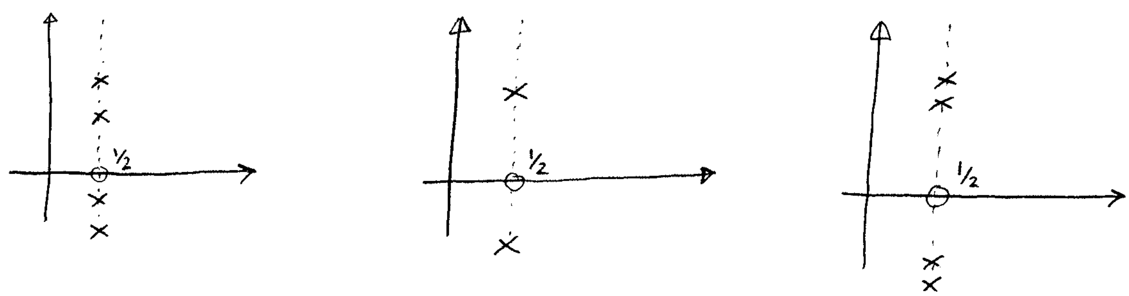
Define $E_d: \; dy^2 = x^3 + ax + b$ "Quadratic twists of E"

What is the distribution of values of $\{r(E_d)\}$ as d varies?

Associate L-functions, $$L(s,E) = \sum_{n=1}^{\infty} \frac{a_n}{n^{s+1/2}}.$$

a_n — solutions of E mod p
— Fourier coefficients of a weight 2 cusp form.

$$L(s,E_d) = \sum_{n=1}^{\infty} \frac{a_n}{n^{s+1/2}} \chi_d(n)$$



o Symmetry points of functional equation.

<u>Birch – Swinnerton-Dyer conjecture</u>:

Order of vanishing of $L(s,E_d)$ at $s=1/2$ = rank of E_d.

According to Katz-Sarnak $\{E_d\}$ is an orthogonal family

Restrict to cases where the root number is +1. These correspond to matrices from $SO(2N)$.

Moments of $L(\frac{1}{2}, E_d)$ $\longleftrightarrow$ moments of $Z(A,0)$ for $A \in SO(2N)$

Value distribution of $L(\frac{1}{2}, E_d)$ $\longleftrightarrow$ Value distribution of $Z(A,0)$ for $A \in SO(2N)$.

$$\mathbb{P}(|Z(A,0)| < x) \asymp \int_0^x w^{-1/2}\,dw \asymp x^{1/2} \quad \text{as } x \to 0$$

i.e this probability vanishes as $x \to 0$.

For arithmetical reasons, values of $L(\frac{1}{2}, E_d)$ are discretised. (Waldspurger, Shimura, ...)

$$L(\tfrac{1}{2}, E_d) = \frac{\kappa\, c(|d|)^2}{\sqrt{|d|}}$$

Fourier coefficient of a $\frac{3}{2}$-weight form. Integers.

(Be careful about $d<0$, $d>0$.)

$$\Rightarrow L(\tfrac{1}{2}, E_d) = 0 \quad \text{if} \quad L(\tfrac{1}{2}, E_d) < \frac{\kappa}{\sqrt{|d|}}$$

So the probability that $L(\frac{1}{2}, E_d) = 0$ (is)?

$$\int_0^{\frac{\kappa}{\sqrt{|d|}}} h(\log|d|)\, w^{-1/2}\,dw$$

[Speculation.]

Mean density argument

This leads to some conjectures:

Conjecture 1

$$\#\{p < x : L(\tfrac{1}{2}, E_{-p}) = 0 \text{ non-trivially}\} \asymp \frac{x^{3/4}}{(\log x)^{5/8}}$$

Conjecture 2

$$\lim_{x\to\infty} \frac{\#\{d < x : L(\tfrac{1}{2}, E_{-d}) = 0 \text{ non-trivially; and } \chi_{-d}(p) = +1\}}{\#\{d < x : L(\tfrac{1}{2}, E_{-d}) = 0 \text{ non-trivially; and } \chi_{-d}(p) = -1\}}$$

$$= \sqrt{\frac{p+1-a_p}{p+1+a_p}}.$$