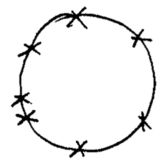
Keating 2

A an $N \times N$ unitary matrix, $A \in U(N)$.

Eigenvalues $e^{i\theta_n}$



Two point correlation function

Let $\phi_n = \dfrac{\theta_n N}{2\pi}$ be re-scaled eigenangles, with mean spacing 1.

$$R_2(A, x) = \frac{1}{N} \sum_{k=-\infty}^{\infty} \sum_{m,n} \delta\left(x - (\phi_n - \phi_m) + kN\right)$$

e.g. if f is a 2π-periodic test function,

$$\sum_{n,m} f(\phi_n - \phi_m) = \int_{-\infty}^{\infty} f(x) R_2(A, x)\, dx.$$

[**Aside**

In physics we usually see

$$d(\phi) = \sum_{k=-\infty}^{\infty} \sum_{n} \delta(\phi - \phi_n + kN)$$

Then

$$R_2(A, x) = \frac{1}{N} \int_0^N d(\phi)\, d(\phi + x)\, d\phi.$$]

Exercise: Show 2 definitions are equivalent.

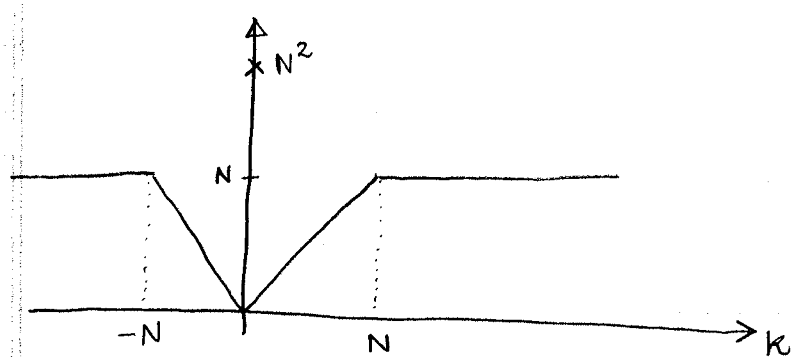
Goal: To compute $\mathbb{E}_{U(N)} R_2(A,x)$.

Write
$$R_2(A,x) = \frac{1}{N^2}\sum_{j=-\infty}^{\infty} |\mathrm{Tr}\, A^j|^2 e^{-2\pi i j x/N}$$
(Exercise: Prove this.)

So look at $\mathbb{E}_{U(N)} |\mathrm{Tr}\, A^j|^2$

Theorem (Dyson)
$$\mathbb{E}_{U(N)} |\mathrm{Tr}\, A^k|^2 = \begin{cases} N^2, & k=0, \\ |k|, & 0<|k|<N, \\ N, & |k|\geq N. \end{cases}$$

Picture



Lemma

For f a class function,
$$\mathbb{E}_{U(N)} f(A) = \frac{1}{(2\pi)^N}\int_0^{2\pi}\cdots\int_0^{2\pi} f(\theta_1,\dots,\theta_N)\det\left(e^{i\theta_n(n-m)}\right)d\theta_1\cdots d\theta_N$$

Proof

$$\mathbb{E}_{U(N)} f = \frac{1}{(2\pi)^N N!} \int_0^{2\pi} \cdots \int_0^{2\pi} f \cdot \prod_{n<m} |e^{i\theta_n} - e^{i\theta_m}|^2 \, d\theta_1 \cdots d\theta_N \quad \text{(Weyl)}$$

$$\det \begin{vmatrix} 1 & 1 & 1 & \cdots \\ e^{i\theta_1} & e^{i\theta_2} & e^{i\theta_3} & \cdots \\ \vdots & \vdots & \vdots & \end{vmatrix} \begin{vmatrix} 1 & e^{-i\theta_1} & e^{-2i\theta_1} & \cdots \\ 1 & e^{-i\theta_2} & e^{-2i\theta_2} & \cdots \\ 1 & e^{-i\theta_3} & \vdots & \\ \vdots & \vdots & & \end{vmatrix}$$

$$= \det \left(\sum_{\ell=1}^N e^{i\theta_\ell (n-m)} \right)$$

$$\mathbb{E}_{U(N)} f = \frac{1}{(2\pi)^N N!} \int_0^{2\pi} \cdots \int_0^{2\pi} f(\theta_1, \ldots, \theta_N) \det \begin{vmatrix} \sum_{\ell=1}^N 1 & \sum_{\ell=1}^N e^{-i\theta_\ell} & \sum_{\ell=1}^N e^{-2i\theta_\ell} \\ \sum_{\ell=1}^N e^{i\theta_\ell} & \sum_{\ell=1}^N 1 & \sum_{\ell=1}^N e^{-i\theta_\ell} \\ \vdots & \vdots & \vdots \end{vmatrix} \times d\theta_1 \cdots d\theta_N.$$

[See Keating notes in book for details of the simplification.]

□

Now,

$$\mathbb{E}_{U(N)} |\mathrm{Tr}\, A^k|^2 = \frac{1}{(2\pi)^N} \int_0^{2\pi} \cdots \int_0^{2\pi} \sum_p \sum_q e^{i(\theta_p - \theta_q)k}$$

$$\times \det \begin{vmatrix} 1 & e^{-i\theta_1} & e^{-2i\theta_1} & \cdots & e^{-(N-2)i\theta_1} & e^{-(N-1)i} \\ e^{i\theta_2} & 1 & e^{-i\theta_2} & \cdots & & e^{-(N-2)} \\ \vdots & & & & & \\ e^{i(N-1)\theta_N} & e^{i(N-2)\theta_N} & & \cdots & & 1 \end{vmatrix}$$

$$\times \, d\theta_1 \cdots d\theta_N$$

Look first at diagonal $(p=q)$ terms. There are N such te
each contributing 1. So contribution is N.

Off-diagonal terms, $(p \neq q)$ Use $\frac{1}{2\pi}\int_0^{2\pi} e^{in\theta}\, d\theta = \delta_n^0$.

If $|k| \geq N$, there are not any powers in the determinant big enough to cancel the $e^{i(\theta_p - \theta_q)k}$.

If $k = N - j$ the contribution is $-j$ from the j terms in the determinant that cancel the other factors.

Back to R_2 ...

$$\mathbb{E}_{U(N)} R_2(A,x) = \sum_{k=-\infty}^{\infty} \begin{Bmatrix} N^2 & k=0 \\ |k| & |k|<N \\ N & |k| \geq N \end{Bmatrix} e^{-2\pi i k x/N}$$

$$= \sum_{j=-\infty}^{\infty} \delta(x - jN) + 1 - \frac{\sin^2 \pi x}{N^2 \sin^2\left(\frac{\pi x}{N}\right)}$$

An exact formula for any N.

For $f(x)$ a test function

$$\lim_{N\to\infty} \mathbb{E}_{U(N)} \int_{-\infty}^{\infty} f(x) R_2(A,x)\, dx = \int_{-\infty}^{\infty} f(x)\left(\delta(x) + 1 - \frac{\sin^2 \pi x}{\pi^2 x^2}\right) dx.$$

Eigenvalues are correlated due to the presence of this term

NB.

$$\mathbb{E}_{U(N)} R_2(A,x) = \sum_{j=-\infty}^{\infty} \delta(x-jN) + \det_{2\times 2} \begin{vmatrix} 1 & \dfrac{\sin \pi x}{N \sin \frac{\pi x}{N}} \\ \dfrac{\sin x\pi}{N \sin \frac{\pi x}{N}} & 1 \end{vmatrix}$$

Generalisations

$$R_n(A, x_1, x_2, \ldots, x_n) = \frac{1}{N}\int_0^N d(\phi+x_1)\, d(\phi+x_2) \cdots \overline{d(\phi+x_n)}\, d\phi$$

$$\mathbb{E}_{U(N)} R_n(A, x_1, \ldots, x_n) = \delta\text{-functions} + \det_{n\times n} \left| \frac{\sin \pi(x_j - x_k)}{N \sin \frac{\pi(x_j-x_k)}{N}} \right|$$

Equivalent definitions

Let $\quad P(\theta_1, \ldots, \theta_N) = \dfrac{1}{(2\pi)^N N!} \prod_{j<k} |e^{i\theta_j} - e^{i\theta_k}|^2$

$$\tilde{R}_2(\theta_1, \theta_2) = N(N-1) \underbrace{\int_0^{2\pi} \cdots \int_0^{2\pi}}_{N-2} P(\theta_1, \theta_2, \theta_3, \ldots, \theta_N)\, d\theta_3 \cdots d\theta_N$$

In a similar way

$$\tilde{R}_n(\theta_1, \ldots, \theta_n) = \frac{N!}{(N-n)!} \underbrace{\int_0^{2\pi} \cdots \int_0^{2\pi}}_{N-n} P(\theta_1, \ldots, \theta_N)\, d\theta_{n+1} \cdots d\theta_N .$$

$$= \det_{n\times n} \left| \frac{1}{2\pi} \frac{\sin(N(\theta_j - \theta_k)/2)}{\sin((\theta_j - \theta_k)/2} \right|$$

$$\left(\frac{2\pi}{N}\right)^n \tilde{R}(\phi_1, \ldots, \phi_n) = \mathbb{E}_{U(N)} R_2(A, \phi_1, \ldots, \phi_n) .$$