Haar measure and Weyl integration

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Slide 1

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Slide 2

The purpose of this talk is to prove Weyl's integration formula for $U(N)$.

Theorem Suppose f is a class function on the unitary group $U(N)$, and let $\mathbb{E}_N$ denote expectation with respect to Haar measure on $U(N)$. Then

$$\mathbb{E}_N[f(A)] = \mathbb{E}_N[f(UAU^{-1})] = \mathbb{E}_N[f(\theta_1, \dots, \theta_n)]$$
$$= \frac{1}{(2\pi)^N N!} \int_{(-\pi,\pi]^N} \prod_{1\le j<k\le N} |e^{\mathrm{i}\theta_k} - e^{\mathrm{i}\theta_j}|^2 f(\theta_1, \dots, \theta_N) \mathrm{d}\theta_1 \dots \mathrm{d}\theta_N$$

[Note abuse of notation for f]

Slide 3

Diversion

The average of a function over $\mathbb{R}^3$ is

$$\int_{\mathbb{R}^3} f(\underline{x})\ \mathrm{d}\underline{x}$$

If f is rotation invariant then $f(\underline{x}) = f(|\underline{x}|)$ [note the abuse of notation!] and by changing to spherical coordinates:

$$\int_{\mathbb{R}^3} f(\underline{x})\ \mathrm{d}\underline{x} = 4\pi \int_0^\infty r^2 f(r)\ \mathrm{d}r$$

- Density vanishes at $r = 0$, which is the origin in $\mathbb{R}^3$
- This is one point in three-space, so has co-dimension 3.
- The density looks like a square.

Slide 4

A class function in $U(N)$ is similar to a rotation invariant function in $\mathbb{R}^3$. It is a function which depends only the eigenvalues of the input.

I.e. $f(A) = f(\theta_1, \ldots, \theta_N)$, or $f(A) = f(UAU^{-1})$ for all unitary U.

We will show that the condition of two or more eigenvalues being the same is a codimension 3 condition.

We will show the Haar measure vanishes like a square on that subspace

Slide 5

Regular Borel measure on a Lie group: Interacts nicely with the locally compact Hausdorff topology, e.g. $\mu(K) < \infty$ for compacts.

A left-Haar measure on a Lie group G is a nonzero regular Borel measure μ on G which is left invariant:

$$\mu(gE) = \mu(E) \text{ for every Borel set} E \subset G, \forall g \in G$$

In other words, if $f \in L^1(G)$ then

$$\int_G f(gA)\mathrm{d}\mu(A) = \int_G f(A)d\mu(A)$$

Every Lie group has a left-Haar measure.

A left-Haar measure which is also a right-Haar measure is a *Haar measure.*

Theorem. Every compact Lie group has a Haar measure. Since compact, can be made into probability measure.

Slide 6

A complex $N \times N$ matrix has $2N^2$ degrees of freedom.

But a unitary matrix satisfies $AA^* = I$, which forces N^2 terms. Therefore a unitary matrix has N^2 real degrees of freedom.

A unitary matrix A can be diagonalized: There exists unitary matrix U such that $A = UDU^{-1}$, with D diagonal.

Eigenvalues $\{e^{\mathrm{i}\theta_1}, \ldots, e^{\mathrm{i}\theta_N}\}$. Note order doesn't matter, any of the possible permutations are allowed (and there are $N!$ if the eigenvalues are distinct).

Slide 7

Given A and D, how many degrees of freedom does U have?

Suppose

$$A = U_1 D U_1^{-1} = U_2 D U_2^{-1}$$

i.e.

$$(U_2^{-1} U_1) D = D (U_2^{-1} U_1)$$

I.e. $V = U_2^{-1} U_1$ commutes with D.

Slide 8

$$\begin{pmatrix} v_{11} & \dots & v_{1N} \\ \vdots & & \vdots \\ v_{N1} & \dots & v_{NN} \end{pmatrix} \begin{pmatrix} e^{\mathrm{i}\theta_1} & & \\ & \ddots & \\ & & e^{\mathrm{i}\theta_N} \end{pmatrix}$$

$$= \begin{pmatrix} e^{\mathrm{i}\theta_1} & & \\ & \ddots & \\ & & e^{\mathrm{i}\theta_N} \end{pmatrix} \begin{pmatrix} v_{11} & \dots & v_{1N} \\ \vdots & & \vdots \\ v_{N1} & \dots & v_{NN} \end{pmatrix}$$

The element in the jth row and kth column is

$$v_{jk} e^{\mathrm{i}\theta_k} = e^{\mathrm{i}\theta_j} v_{jk}$$

so if eigenvalues are distinct, $v_{jk} = 0$ for $j \neq k$, i.e. V must be diagonal, and so $U \in U(N)/\mathrm{diag} =: [U]$ ranges over an $N^2 - N$ dim set. So A (with distinct eigenvalues) has N^2 dimensions.

Slide 9

What happens if $e^{i\theta_1} = e^{i\theta_2}$?

Then V is of the form

$$\left(\begin{array}{cc|c} v_{11} & v_{12} & \\ v_{21} & v_{22} & \\ \hline & & \text{diagonal} \end{array}\right)$$

$U(N)/(\text{matrices of shape } V)$ has $N^2 - (N+2)$ dimensions.

D has $N-1$ dimensions (since two evals the same)

So the submanifold of $U(N)$ with two evals the same has $N^2 - (N+2) + (N-1) = N^2 - 3$ dimensions.

I.e. is of codimension 3.

Slide 10

Since $e^{i\theta_1} = e^{i\theta_2}$ is a codimension 3 condition, we expect the density to vanish like $|e^{i\theta_2} - e^{i\theta_1}|^2$. This must happen for all pairs θ_j, θ_k, so expect a factor of

$$\prod_{1 \le j < k \le N} |e^{i\theta_k} - e^{i\theta_j}|^2$$

Now let's prove that.

Assume the measure is invariant under rotation (Haar)

Assume A has distinct eigenvalues.

Slightly wiggle A; what effect does this have on the eigenvalues.

Slide 11

Select a basis for the N^2 dimensional tangent space of $U(N)$ at I. The tangent space is given by wanting

$$(I + \epsilon B)^*(I + \epsilon B) = I$$

i.e. $B^* + B = 0$ to leading order, so anti-hermitian, so independent values $\mathfrak{Im}(B_{ii})$, $\mathfrak{Re}(B_{ij})$, $\mathfrak{Im}(B_{ij})$ for $i < j$.

[[diversion on Lie algebra??]]

From A move to $A + \mathrm{d}A = A(I + A^{-1}\mathrm{d}A)$, so $\epsilon B = \delta A = A^{-1}\mathrm{d}A$.

Slide 12

$$A = UDU^{-1} \Longrightarrow AU = UD$$

Take differentials

$$(\mathrm{d}A)U + A\mathrm{d}U = (\mathrm{d}U)D + U\mathrm{d}D$$

Left-multiply by

$$U^{-1}A^{-1} = D^{-1}U^{-1}$$

to get

$$U^{-1}A^{-1}(\mathrm{d}A)U + U^{-1}A^{-1}A\mathrm{d}U = D^{-1}U^{-1}(\mathrm{d}U)D + D^{-1}U^{-1}U\mathrm{d}D$$

If $\delta A = A^{-1}\mathrm{d}A$, $\delta U = U^{-1}\mathrm{d}U$, $\delta D = D^{-1}\mathrm{d}D$, this equals

$$U^{-1}(\delta A)U + (\delta U) = D^{-1}(\delta U)D + (\delta D)$$

Slide 13

$$D = \begin{pmatrix} e^{\mathrm{i}\theta_1} & & \\ & \ddots & \\ & & e^{\mathrm{i}\theta_N} \end{pmatrix}$$

$$\mathrm{d}D = \begin{pmatrix} \mathrm{i}e^{\mathrm{i}\theta_1}\mathrm{d}\theta_1 & & \\ & \ddots & \\ & & \mathrm{i}e^{\mathrm{i}\theta_N}\mathrm{d}\theta_N \end{pmatrix}$$

$$\delta D = D^{-1}\mathrm{d}D = \begin{pmatrix} \mathrm{id}\theta_1 & & \\ & \ddots & \\ & & \mathrm{id}\theta_N \end{pmatrix}$$

Slide 14

Write $U^{-1}(\delta A)U = (\delta b_{jk})$, and $\delta U = (\delta u_{jk})$. From

$$U^{-1}(\delta A)U = D^{-1}(\delta U)D + (\delta D) - (\delta U)$$

and so

$$\delta b_{jj} = \mathrm{id}\theta_{jj}$$

and for $j \neq k$,

$$\begin{aligned} \delta b_{jk} &= e^{-\mathrm{i}\theta_j}\delta u_{jk} e^{\mathrm{i}\theta_k} - \delta u_{jk} \\ &= \left(\frac{e^{\mathrm{i}\theta_k}}{e^{\mathrm{i}\theta_j}} - 1\right)\delta u_{jk} \end{aligned}$$

Note can take $\delta u_{jj} = 0$.

Slide 15

$$\begin{pmatrix} \mathfrak{Re}(\delta b_{jk}) \\ \mathfrak{Im}(\delta b_{jk}) \end{pmatrix} = \begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix} \begin{pmatrix} \mathfrak{Re}(\delta u_{jk}) \\ \mathfrak{Im}(\delta u_{jk}) \end{pmatrix}$$

where $\alpha = \mathfrak{Re}\left(\frac{e^{\mathrm{i}\theta_k}}{e^{\mathrm{i}\theta_j}} - 1\right)$ and $\beta = \mathfrak{Im}\left(\frac{e^{\mathrm{i}\theta_k}}{e^{\mathrm{i}\theta_j}} - 1\right)$

Jacobian: $|\det| = \alpha^2 + \beta^2 = \left|\frac{e^{\mathrm{i}\theta_k}}{e^{\mathrm{i}\theta_j}} - 1\right|^2$

Slide 16

Since Haar measure, $\mu(A) = \mu(U^{-1}AU)$.

Therefore, the volume element $\mathrm{d}\mu(A)$ around A becomes

$$\begin{aligned} \mathrm{d}\mu(A) &= \prod_{1\le j<k\le N} \left|\frac{e^{\mathrm{i}\theta_k}}{e^{\mathrm{i}\theta_j}} - 1\right|^2 \cdot [\mathrm{d}\omega_U] \cdot \mathrm{d}\theta_1 \cdots \mathrm{d}\theta_N \\ &= \prod_{1\le j<k\le N} |e^{\mathrm{i}\theta_k} - e^{\mathrm{i}\theta_j}|^2 [\mathrm{d}\omega_U] \cdot \mathrm{d}\theta_1 \cdots \mathrm{d}\theta_N \end{aligned}$$

where $[\mathrm{d}\omega_U]$ represents the volume element of $U \in [U(N)]$ where $[U(N)] = U(N)/\Lambda(N)$ where $\Lambda(N) = \{\text{diag unitary}\}$, and comes from $\mathfrak{Re}(\delta u_{jk})$ and $\mathfrak{Im}(\delta u_{jk})$ with $j < k$.

If f is trace class this gets integrated out, so we don't actually care about the structure of $[\mathrm{d}\omega_U]$.

Slide 17

For the other groups:

- Symplectic ($AJA^t = J$, $J = \begin{pmatrix} 0 & I_N \\ -I_N & 0 \end{pmatrix}$).

$$\frac{2^{N^2}}{\pi^N N!} \prod_{1 \le j < k \le N} (\cos(\theta_k) - \cos(\theta_j))^2 \prod_{n=1}^{N} \sin^2(\theta_n)$$

- Even orthogonal ($AA^t = I$, matrix of size $2N$)

$$\frac{2^{(N-1)^2}}{\pi^N N!} \prod_{1 \le j < k \le N} (\cos(\theta_k) - \cos(\theta_j))^2$$

- Odd orthogonal ($AA^t = I$, matrix of size $2N + 1$)

$$\frac{2^{N^2}}{\pi^N N!} \prod_{1 \le j < k \le N} (\cos(\theta_k) - \cos(\theta_j))^2 \prod_{n=1}^{N} \sin^2(\tfrac{1}{2}\theta_n)$$