MOMENTS III

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1. Moments and Zeros

Mean value estimates are very useful for studying the zeros of the zeta function; this is one of the reasons so much effort has been expended on them. One link between means and zeros can be seen in Jensen's Formula from classical function theory.

Theorem 1. (Jensen's Formula) *Let $f(z)$ be analytic for $|z| \le R$ and suppose that $f(0) \ne 0$. If $r_1, r_2, \ldots, r_n$ are the moduli of all the zeros of $f(z)$ inside $|z| \le R$, then*

$$\log\left(\frac{|f(0)|R^n}{r_1 r_2 \cdots r_n}\right) = \frac{1}{2\pi}\int_0^{2\pi} \log|f(Re^{i\theta})|\, d\theta\,.$$

Here we see that the mean value of $\log|f(z)|$ around a circle is related to the distribution of the zeros of $f(z)$ inside that circle. There is an analogous result for rectangles, which is often more useful when working with Dirichlet series, namely,

Theorem 2. (Littlewood's Lemma) *Let $f(s)$ be analytic and nonzero on the rectangle $\mathcal{C}$ with vertices σ_0, σ_1, $\sigma_1 + iT$, and $\sigma_0 + iT$, where $\sigma_0 < \sigma_1$. Then*

$$2\pi \sum_{\rho \in \mathcal{C}} \mathit{Dist}(\rho) = \int_0^T \log|f(\sigma_0 + it)|\, dt - \int_0^T \log|f(\sigma_1 + it)|\, dt$$

$$+ \int_{\sigma_0}^{\sigma_1} \arg f(\sigma + iT)\, d\sigma - \int_{\sigma_0}^{\sigma_1} \arg f(\sigma)\, d\sigma\,,$$

where the sum runs over the zeros ρ of $f(s)$ in $\mathcal{C}$ and "Dist(ρ)" is the distance from ρ to the left edge of the rectangle.

When we use Littlewood's Lemma below, only the first term on the right–hand side will be significant. In order not to be too technical, I will always use the result in the form

$$2\pi \sum_{\rho \in \mathcal{C}} \mathrm{Dist}(\rho) = \int_0^T \log|f(\sigma_0 + it)|\, dt + E,$$

where E is an error term that can be ignored and may be different on different occassions.

The integral of the logarithm usually cannot be dealt with directly, so we often use the following trick. We have

$$\frac{1}{T}\int_0^T \log|f(\sigma+it)|\,dt = \frac{1}{2T}\int_0^T \log(|f(\sigma+it)|^2)\,dt \le \frac{1}{2}\log(\int_0^T |f(\sigma+it)|^2\,dt)\,,$$

where the inequality follows from the arithmetic–geometric mean inequality. Now we see a direct connection between the location of the zeros within a rectangle and the type of mean–values we have been considering *via* Littlewood's Lemma.

2. Application I: A Simple Zero–Density Estimate

We want to show that there are relatively few zeros of the zeta–function in the right half of the critical strip. Let σ_0 be a fixed real number strictly between 1/2 and 1 and let $\mathcal{C}$ be the rectangle in the complex plane with vertices at 2, $2+iT$, σ_0+iT, σ_0. Applying our (simplified) version of Littlewood's Lemma, we see that

$$\sum_{\rho\in\mathcal{C}} \mathrm{Dist}(\rho) = \frac{1}{2\pi}\int_0^T \log(|\zeta(\sigma_0+it)|)\,dt + E,$$

where $\mathrm{Dist}(\rho)$ is the distance of the zero ρ from the line $Re\ s = \sigma_0$. Now let σ be a fixed real number with $\sigma_0 < \sigma < 1$ and write $N(\sigma, T)$ for the number of zeros $\rho = \beta + i\gamma$ of $\zeta(s)$ with $\sigma < \beta \le 2$ and $0 < \gamma < T$. On the one hand, we have

$$\sum_{\rho\in\mathcal{C}} \mathrm{Dist}(\rho) \ge \sum_{\substack{\rho\in\mathcal{C}\\ \sigma\le\beta}} \mathrm{Dist}(\rho) \ge (\sigma-\sigma_0)N(\sigma,T).$$

On the other hand,

$$\begin{aligned}\frac{1}{2\pi}\int_0^T \log(|\zeta(\sigma_0+it)|)\,dt &= \frac{1}{4\pi}\int_0^T \log(|\zeta(\sigma_0+it)|^2)\,dt \\ &\le \frac{T}{4\pi}\log(\frac{1}{T}\int_0^T |\zeta(\sigma_0+it)|^2)\,dt\end{aligned}$$

by the arithmetic–geometric mean inequality, as before. The integral on the last line is $I_k(\sigma_0, T)$, which we have seen is $\sim \zeta(2\sigma_0)T$. Thus, the last expression is $O(T)$. It follows that

$$N(\sigma, T) \ll T\,.$$

Since $N(T) \sim \frac{T}{2\pi} \log T$, we see that

$$N(\sigma, T)/N(T) = O(\frac{1}{\log T})$$

for any fixed $\sigma > 1/2$. We may interpret this as saying that the proportion of zeros to the right of any line $\Re s = \sigma > 1/2$ is infinitesimal.

This, the first zero–density estimate, was proved by H. Bohr and E. Landau in 1914. Since then much stronger results have been proved, typically of the form

$$N(\sigma, T) << T^{\lambda(\sigma)} \,,$$

where $\lambda(\sigma) < 1$ for $\sigma > 1/2$. Nevertheless, the underlying idea in the proof of many of these results already appears here.

3. Application II: Levinson's Method

Zero–density theorems tell us there are (relatively) few zeros to the right of the critical line. Our goal here is to sketch the method of Levinson, which shows that there are many zeros *on* it.

Recall that

$$\begin{aligned} N(T) &= \#\{\rho = \beta + i\gamma \mid \zeta(\rho) = 0, \quad 0 < \gamma < T\} \\ &\sim \frac{T}{2\pi} \log T \end{aligned}$$

and let

$$N_0(T) = \# \left\{ \rho = \frac{1}{2} + i\gamma \mid \zeta(\rho) = 0, \quad 0 < \gamma < T \right\}$$

denote the number of zeros on the critical line up to height T. The important estimations of $N_0(T)$ were:

G. H. Hardy (1914) : $N_0(T) \to \infty$ (as $T \to \infty$)

G. H. Hardy-J. E. Littlewood (1921) : $N_0(T) > cT$

A. Selberg (1942) : $N_0(T) > c' N(T)$

N. Levinson (1974) : $N_0(T) > \frac{1}{3} N(T)$

J. B. Conrey (1989) : $N_0(T) > \frac{2}{5} N(T)$

Levinson's method begins with the following fact first proved by Speiser.

Theorem 3. (Speiser) *The Riemann Hypothesis is equivalent to the assertion that $\zeta'(s)$ does not vanish in the left half of the critical strip.*

In the early seventies, N. Levinson and H. L. Montgomery proved a quantitative version of this. Let

$$N'_{-}(T) = \#\{\rho' = \beta' + i\gamma' \mid \zeta'(\rho') = 0,\ -1 < \beta' < 1/2, \quad 0 < \gamma' < T\}$$

and

$$N_{-}(T) = \#\{\rho = \beta + i\gamma \mid \zeta(\rho) = 0,\ -1 < \beta < 1/2, \quad 0 < \gamma < T\}\ .$$

Theorem 4. (Levinson-Montgomery) *We have* $N_{-}(T) = N'_{-}(T) + O(\log T)$.

The idea behind the proof is as follows. Let $0 < a < 1/2$ and let $\mathcal{C}$ denote the positively oriented rectangle with vertices $a + iT/2$, $a + iT$, $-1 + iT$, and $-1 + iT/2$. By a standard method it is not difficult to show that

$$\Delta \arg \frac{\zeta'}{\zeta}(s)\bigg|_{\mathcal{C}} = O(\log T),$$

independently of a. Given this, we see that

$$2\pi(\#\text{ zeros of } \zeta'(s) \text{ in } \mathcal{C} - \#\text{ zeros of } \zeta(s) \text{ in } \mathcal{C}) = O(\log T).$$

The theorem now follows on observing that a was arbitrary, and by "adding" rectangles with top and bottom edges, respectively, at T and $T/2$, $T/2$ and $T/4$, $\dots$.

We now sketch Levinson's method. We have just seen that $N_{-}(T) = N'_{-}(T) + O(\log T)$. Now, the nontrivial zeros of $\zeta(s)$ are symmetric about the critical line. Hence, the number of them lying to the right of the critical line, to the left of the line $\sigma = 2$, and above the real axis up to height T is also $N_{-}(T)$. Therefore

$$\begin{aligned} N(T) &= N_0(T) + 2N_{-}(T) \\ &= N_0(T) + 2N'_{-}(T) + O(\log T), \end{aligned}$$

or

$$N_0(T) = N(T) - 2N'_-(T) + O(\log T).$$

The size of the first term on the left hand side of the last line is known, namely, $(1+o(1))\frac{T}{2\pi}\log T$. Hence, if we can determine a sufficiently small upper bound for $N'_-(T)$, we can deduce a lower bound for $N_0(T)$.

To find such an upper bound it is convenient to first note that the zeros of $\zeta'(s)$ in the region $-1 < \sigma < 1/2,\ 0 < t < T$, are identical to the zeros of $\zeta'(1-s)$ in the reflected region $1/2 < \sigma < 2,\ 0 < t < T$. One can also show, by the functional equation of the zeta–function, that $\zeta'(1-s)$ and $G(s) = \zeta(s) + \zeta'(s)/L(s)$, where $L(s)$ is essentially $\frac{1}{2\pi}\log T$, have the same zeros in $1/2 < \sigma < 2,\ 0 < t < T$. It turns out to be technically advantageous to count the zeros of $G(s)$ rather than those of $\zeta'(1-s)$.

To bound the number of zeros of $G(s)$ in this region, we apply Littlewood's Lemma. Let $a = \frac{1}{2} - \frac{\delta}{\log T}$, with δ a small positive number, and let $\mathcal{R}_a$ denote the rectangle whose vertices are at $a, 2, 2+iT$, and $a+iT$. It would be natural to apply our abreviated form of the lemma to obtain

$$\sum_{\rho^* \in \mathcal{R}_a} \mathrm{Dist}(\rho^*) = \frac{1}{2\pi}\int_0^T \log|G(a+it)|dt + \mathcal{E},$$

where ρ^* denotes a zero of $G(s)$ and $\mathrm{Dist}(\rho^*)$ is its distance to the left edge of $\mathcal{R}_a$. However, in the next step, when we apply the arithmetic–geometric mean inequality to the integral, we would lose too much. To avoid this loss, we first dampen, or mollify, $G(s)$ and apply Littlewood's Lemma in the form

$$\sum_{\substack{\rho^{**} \in \mathcal{R}_a \\ GM(\rho^{**})=0}} \mathrm{Dist}(\rho^{**}) = \frac{1}{2\pi}\int_0^T \log|G(a+it)M(a+it)|dt + \mathcal{E}\,.$$

Here

$$M(s) = \sum_{n \le T^\theta} \frac{a_n}{n^s}\,, \qquad a_n = \mu(n) n^{a-1/2}\left(1 - \frac{\log n}{\log T^\theta}\right)\,,$$

approximates $1/\zeta(s)$ and $\theta > 0$. Note that included among the zeros of $G(s)M(s)$ in $\mathcal{R}_a$ are all the zeros of $G(s)$ in $\mathcal{R}_a$. Therefore we have

$$\begin{aligned}\sum_{\substack{\rho^{**}\in\mathcal{R}_a\\ GM(\rho^{**})=0}} \mathrm{Dist}(\rho^{**}) \geq & \sum_{\substack{\rho^{*}\in\mathcal{R}_a\\ G(\rho^{*})=0}} \mathrm{Dist}(\rho^{*}) \\ \geq & \sum_{\substack{\rho^{*}\in\mathcal{R}_a, \Re\rho^*>1/2\\ G(\rho^{*})=0}} \mathrm{Dist}(\rho^{*}) \\ \geq & (\frac{1}{2}-a)N'_{-}(T)\,.\end{aligned}$$

We now see that

$$\begin{aligned}(1/2-a)N'(T) &\leq \frac{1}{2\pi}\int_0^T \log|GM(a+it)|dt + \mathcal{E} \\ &= \frac{1}{4\pi}\int_0^T \log|GM(a+it)|^2 dt + \mathcal{E} \\ &\leq \frac{T}{4\pi}\log\left(\frac{1}{T}\int_0^T |GM(a+it)|^2 dt\right) + \mathcal{E}\,.\end{aligned}$$

Thus, we require an estimate for

$$\int_0^T |GM(a+it)|^2 dt\,.$$

This is similar to a mean–value we saw in the previous lecture. Levinson was able prove an asymptotic estimate for this integral when $\theta = 1/2-\epsilon$ with ϵ arbitrarily small. The resulting upper bound for $N'_{-}(T)$ then led to the lower bound

$$N_0(T) > \left(\frac{1}{3}+o(1)\right)N(T).$$

Much later, Conrey was able to establish an asymptotic estimate when $\theta = 4/7-\epsilon$, which led to

$$N_0(T) > \left(\frac{2}{5}+o(1)\right)N(T).$$

The form of the asymptotic estimate in both cases is the same as a function of θ, and D. Farmer [F] has given various heuristic arguments that suggest it should remain true even when one takes θ arbitrarily large. From Farmer's conjecture it follows that

$$N_0(T) \sim N(T)\ .$$

Before concluding this section, we remark that had we used mollifiers in our proof of the Bohr–Landau result in the previous section, we could have obtained a much stronger zero–density estimate.

4. Application III: The Number of Simple Zeros

Our next application demonstrates the use of discrete mean value theorems.

Let

$$N_s(T) = \#\{\rho = \beta + i\gamma \mid \zeta(\rho) = 0, \zeta'(\rho) \neq 0, \quad 0 < \gamma < T\}$$

denote the number of simple zeros of the zeta function in the critical strip with ordinates between 0 and T. It is believed that all the nontrivial zeros are on the critical line and simple, in other words, that $N(T) = N_0(T) = N_s(T)$ for every $T > 0$. In 1973, H. Montgomery, used his pair correlation method to show that if the Riemann Hypothesis is true, then at least 2/3 of the zeros are simple. In other words,

$$N_s(T)/N(T) > 2/3$$

provided that T is sufficiently large. We will present his argument in the third lecture. Now, however, we briefly describe a different method of Conrey, Ghosh, and Gonek, which shows that on the stronger hypotheses of RH and the Generalized Lindeloff Hypothesis, one can replace the 2/3 above by $19/27 = .703\dots$.

By the Cauchy–Schwarz inequality, we have

$$\Big|\sum_{0<\gamma<T} \zeta'(\rho)M_N(\rho)\Big|^2 \leq \Big(\sum_{\substack{0<\gamma\leq T\\ \rho+i\gamma\ \text{is simple}}} 1\Big)\Big(\sum_{0<\gamma<T} |\zeta'(\rho)M_N(\rho)|^2\Big),$$

where $M_N(s)$ is a Dirichlet polynomial of length N with coefficients similar, but not identical, to those of $M(s)$ in the last section. Its purpose is also similar: to mollify $\zeta'(\rho + i\gamma)$ so as to minimize the loss in applyng the Cauchy–Schwarz inequality. If one assumes RH, the sum on the left–hand side is easy to compute and turns out to be $\sim \frac{19}{24}N(T)\log T$. The sum on the right–hand side is much more difficult to treat, but one can show that if RH and

GLH are true, then it is $\sim \frac{57}{64}N(T)\log^2 T$. Inserting these estimates into the inequality above and solving for $N_s(T)$ leads to the result stated. An elaboration of the method leads to the conclusion that, on the same hypotheses, at least 95.5% of the zeros of $\zeta(s)$ are either simple or double.

5. Application IV: The Möbius Function

So far all our applications of mean values have been to study the zeros. Our last example is an arithmetical application.

Let

$$M(x) = \sum_{n \le x} \mu(n)$$

be the summatory function for the Möbius function. recall $\mu(n) = (-1)^k$ if $n = p_1 p_2 \dots p_k$ is a product of k distinct primes, and $\mu(n) = 0$ if the square of any prime divides n. It is known that the Riemann Hypothesis is equivalent to the assertion that

$$M(x) \ll x^{1/2+\epsilon}.$$

(Interpretation: the $\mu(n)$ behave like independent random variables.) There is an explicit formula relating $M(x)$ to the zeros of the zeta function that reads

$$M(x) = \sum_{0<\gamma\le T} \frac{x^\rho}{\rho\zeta'(\rho)} + E(x,T).$$

The sum is over the zeros of the zeta function and $E(x,T)$ is an error term. Here we are assuming the zeros are all simple but not the Riemann Hypothesis. There is a more complicated version if the zeros are not all simple involving higher derivatives of $\zeta(\rho)$ and log's in the numerator for such zeros. Let's assume the zeros are simple though, and that RH holds. Then this becomes

$$M(x) = x^{1/2} \sum_{0<\gamma\le T} \frac{x^{i\gamma}}{\rho\zeta'(\frac{1}{2}+i\gamma)} + E(x,T).$$

One can show that the error term can be ignored if $x = T$. Let us assume this. Then we see that

$$|M(x)| \ll x^{1/2} \sum_{0<\gamma\le x} \frac{1}{|\rho\zeta'(\frac{1}{2}+i\gamma)|}\,. \tag{5.1}$$

Hejhal and Gonek independently conjectured that

$$T\log^{1/4} T \ll \sum_{0<\gamma\le T} \frac{1}{|\zeta'(\rho)|} \ll T\log^{1/4} T\,.$$

Partial summation using the upper bound gives

$$\sum_{0<\gamma\le x} \frac{1}{|\rho\zeta'(\rho)|} \ll \log^{5/4} x\,.$$

Thus, we expect that

$$M(x) \ll x^{\frac{1}{2}} \log^{\frac{5}{4}} x\ .$$

This was first conjectured by Gonek (unpublished). N. Ng has a nice paper working out this and other consequences.