Maass forms, the geometry of $\Gamma_0(N)$, L-functions, quantum chaos, ...

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AIM

Intro. to the arithmetic theory of automorphic functions (AMS)
by Iwaniec

Analytic Number Theory (AMS)
by Iwaniec & Kowalski

Bump's book on modular forms

Recall: $SL(2,\mathbb{R})$ acts on $\mathcal{H} = \{x+iy : y>0\}$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}(z) = \frac{az+b}{cz+d}$$

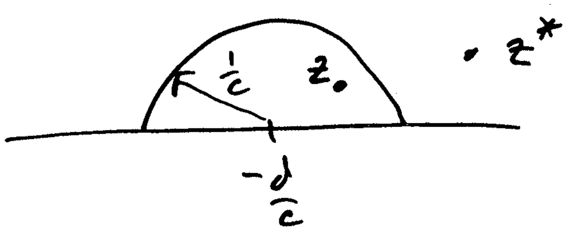
(hyperbolic) isometry

(Projectivize the action of $SL(2,\mathbb{R})$ on $\mathbb{C}^2$:

$$\begin{pmatrix} u \\ v \end{pmatrix} \to \begin{pmatrix} u/v \\ 1 \end{pmatrix}$$
)

$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ maps $\left| z + \frac{d}{c} \right| < \frac{1}{c}$ "higher up"

$z^* = \gamma z$



$\rightsquigarrow$ Highest point fundamental domain for Γ

fold it up to get the Riemann Surface $\mathcal{H}/\Gamma$

R_y© Recall

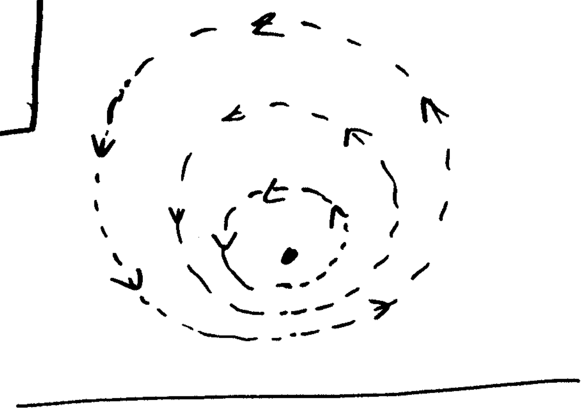
3 kinds of rigid motions.

Fixed points : $\dfrac{az+b}{cz+d} = z$

Elliptic

$|a+d| < 2$

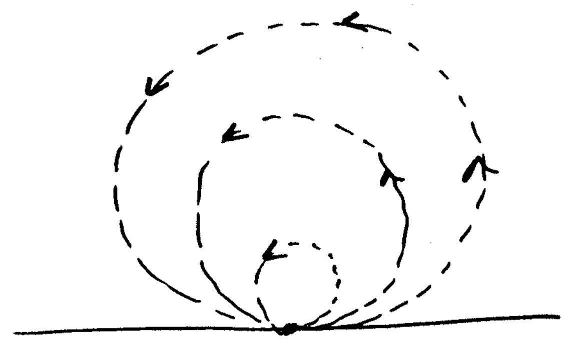
1 fixed point in $\mathcal{H}$



Parabolic

$|a+d| = 2$

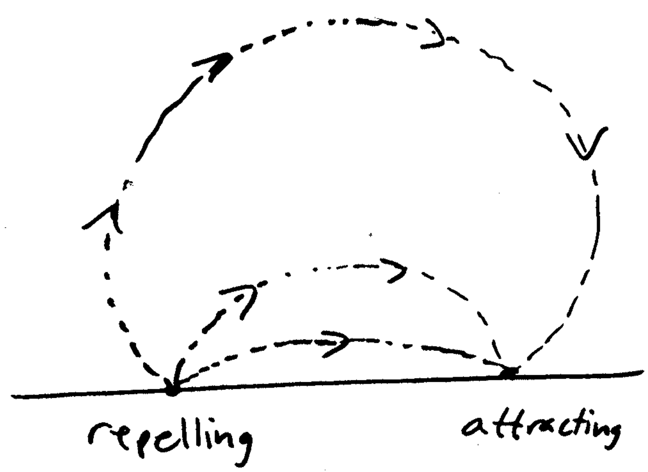
1 fixed point at ∞



Hyperbolic

$|a+d| > 2$

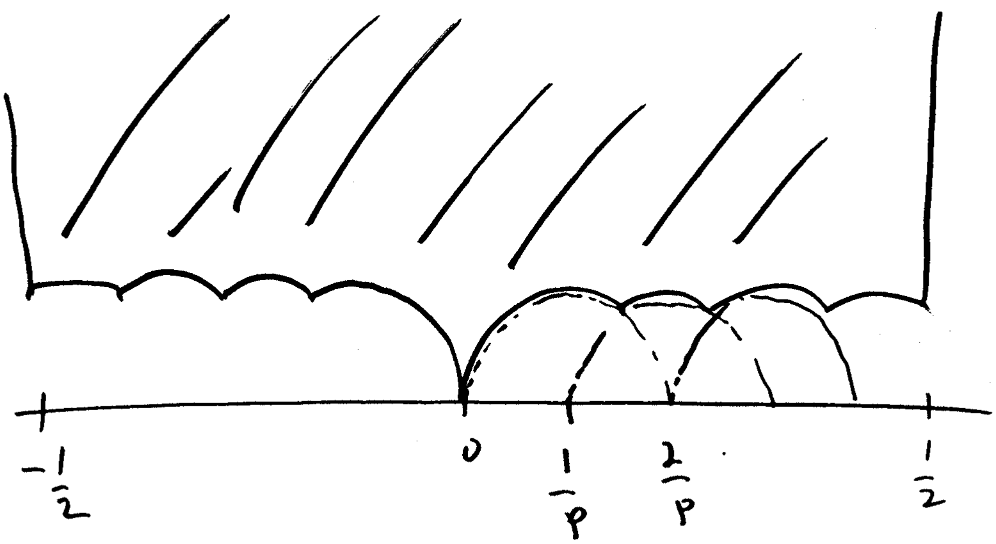
2 fixed points at ∞



Algebra		Geometry
$\Gamma < SL(2,\mathbb{R})$ discrete subgroup		$\mathcal{H}/\Gamma$ Riemann surface
elliptic $\gamma \in \Gamma$	conjugacy class	cone point
parabolic		cusp
hyperbolic		closed geodesic
outer automorphism		symmetry
subgroup		covering space

You can read off the generators and relations of the group from the fundamental domain.

$\Gamma_0(p)$, p prime, has 2 cusps: $0, \infty$

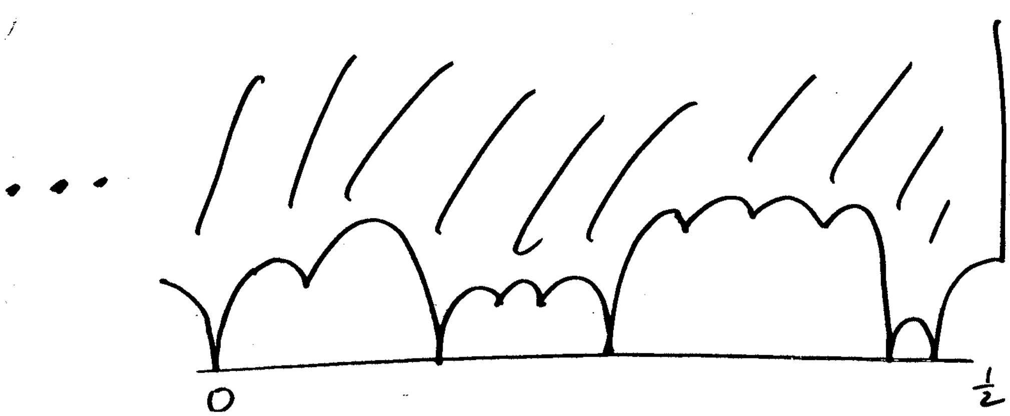


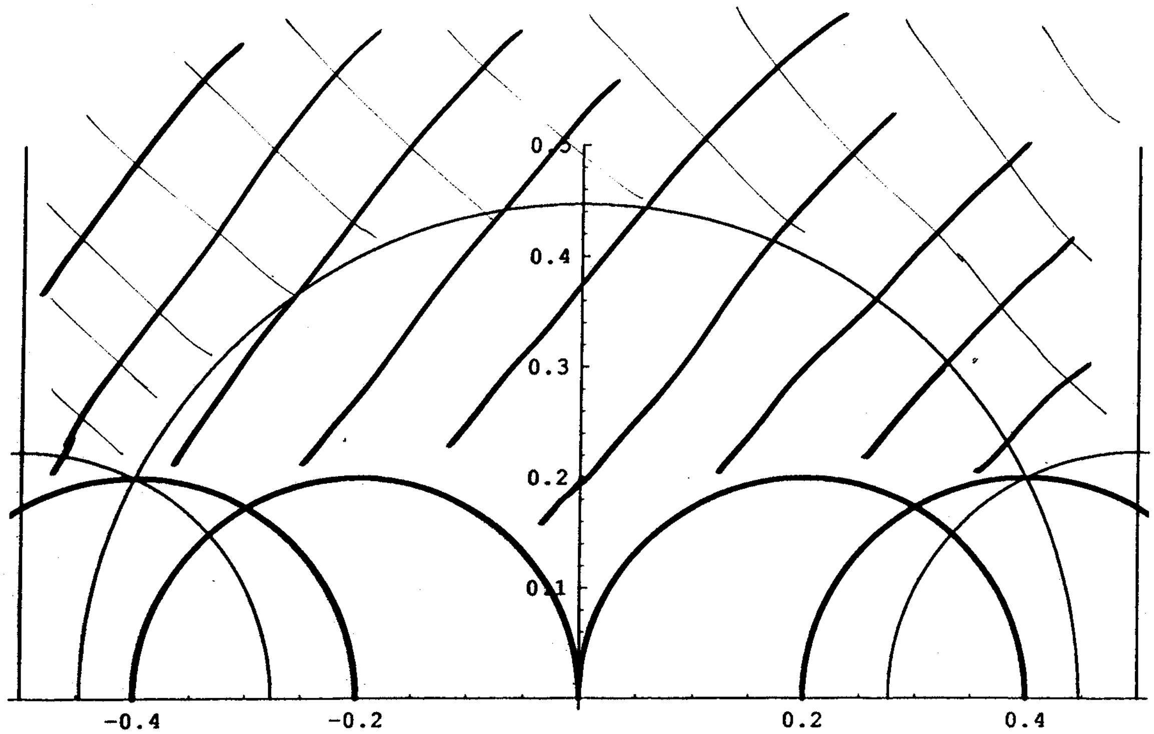
<u>Corollary</u>: $\Gamma_0(p)$ is generated by $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$

and $\begin{pmatrix} a & b \\ p & c \end{pmatrix}$ with $|c| < p/2$

(not a minimal generating set)

$\Gamma_0(N)$ can have a complicated fundamental domain



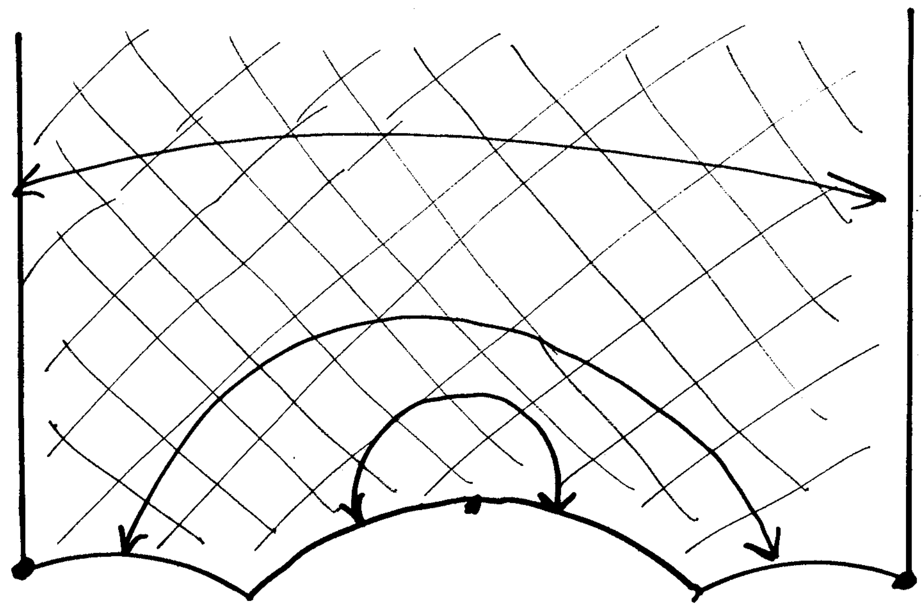


F.D. for $\underline{\Gamma_0(5)}$ and $\underline{\langle \Gamma_0(5), H_5 \rangle}$

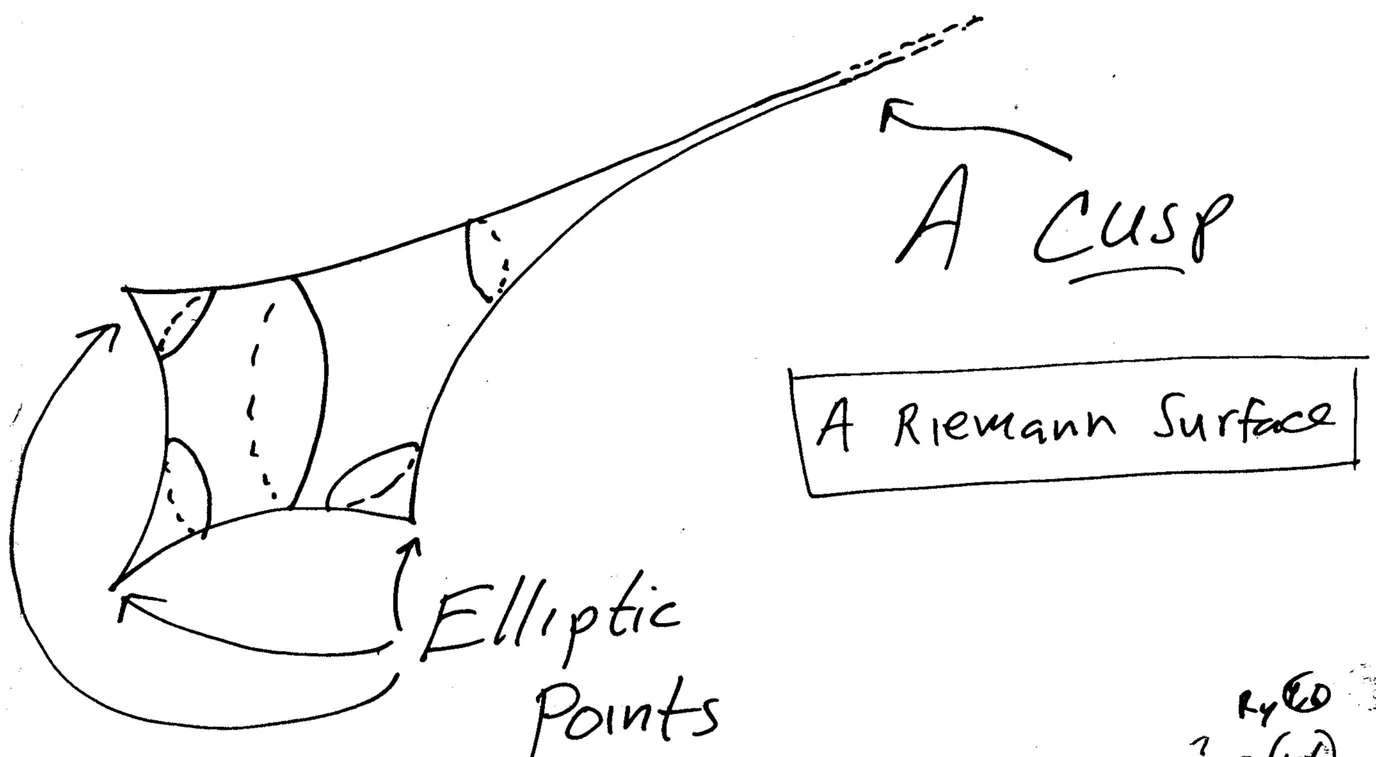
genus 0
2 cusps
2 cone points of order 2

genus 0
1 cusp
3 cone points of order 2

R_Y ⑥

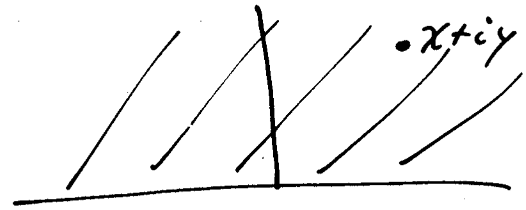


FOLD UP THE FUNDAMENTAL DOMAIN



What is a Maass form?

$f : \mathcal{H} \to \mathbb{R}$



$\Gamma \subset SL(2,\mathbb{R})$

$\mathcal{H} = \{x+iy : y>0\}$

① $f\left(\frac{az+b}{cz+d}\right) = f(z)$ for all $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$

② $\Delta f = \lambda f$ $\quad \Delta = -y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)$

$\lambda = \frac{1}{4} + R^2$

③ f vanishes at all cusps of Γ

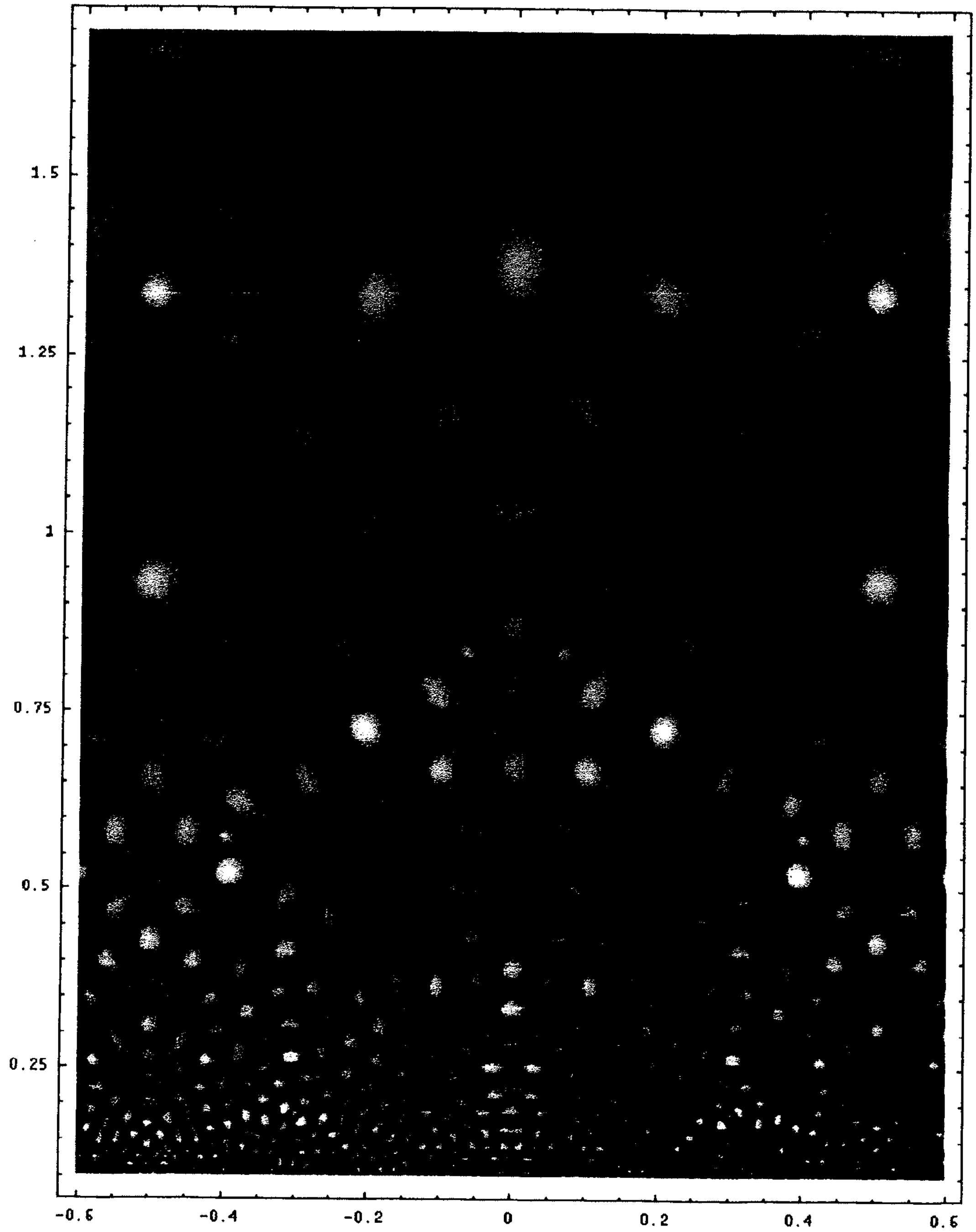
$f(z+1) = f(z)$ if $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in \Gamma$

f has a Fourier expansion with $a_0 = 0$

$\Gamma_0(N)$ have Fricke & Hecke operators

$L(2)$

$F_L(2)$



Level 17, $R \approx 44$

Discrete spectrum $0 < \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \cdots$

Maass (cusp) forms $u_1, u_2, \cdots$

Continuous spectrum $[\frac{1}{4}, \infty)$

Eisenstein series (one per cusp) $E_\mathfrak{a}$

Can't (in general) multiply maass forms

$\langle \cdot, \cdot \rangle$ Petersson inner product (as in Rubinstein's talk)

Spectral decomposition: If $f \in L^2(\mathcal{H}/\Gamma_0(N))$ then

$$f(z) = \sum_{j \geq 0} \langle f, u_j \rangle u_j$$

$$+ \sum_{\text{cusps } \mathfrak{a}} \frac{1}{4\pi} \int_{-\infty}^{\infty} \langle f, E_\mathfrak{a}(\cdot, \tfrac{1}{2}+it) \rangle E_\mathfrak{a}(z, \tfrac{1}{2}+it)\, dt$$

Weil's explicit formula

$$\sum_{\substack{\gamma \\ \text{zeros}}} \phi(\gamma) = \phi\left(\tfrac{i}{2}\right) + \phi\left(\tfrac{-i}{2}\right)$$

$$+ \frac{1}{2\pi} \int_{-\infty}^{\infty} \phi(t) \left(-\log \pi + \operatorname{Re} \frac{\Gamma'}{\Gamma}\left(\tfrac{1}{4} + it\right)\right) dt$$

$$- \frac{1}{2\pi} \sum_{n} \frac{\Lambda(n)}{\sqrt{n}} \left(\hat{\phi}\left(\frac{\log n}{2\pi}\right) + \hat{\phi}\left(-\frac{\log n}{2\pi}\right)\right)$$

prime powers

Selberg trace formula

$$\sum_{\substack{j \\ \text{eigenvalues}}} \phi(R_j) - \frac{1}{4\pi} \int_{-\infty}^{\infty} \phi(r) \frac{\varphi'}{\varphi}\left(\tfrac{1}{2} + ir\right) dr$$

$$= \frac{|F|}{4\pi} \int_{-\infty}^{\infty} \phi(r) \tanh(\pi r)\, dr - \frac{h}{2\pi} \int_{-\infty}^{\infty} \phi(r) \frac{\Gamma'}{\Gamma}(1 + ir)\, dr$$

$$+ 2 \sum_{P} \sum_{\ell=1}^{\infty} \left(p^{\ell/2} - p^{-\ell/2}\right) \hat{\phi}(\ell \log p) \log p$$

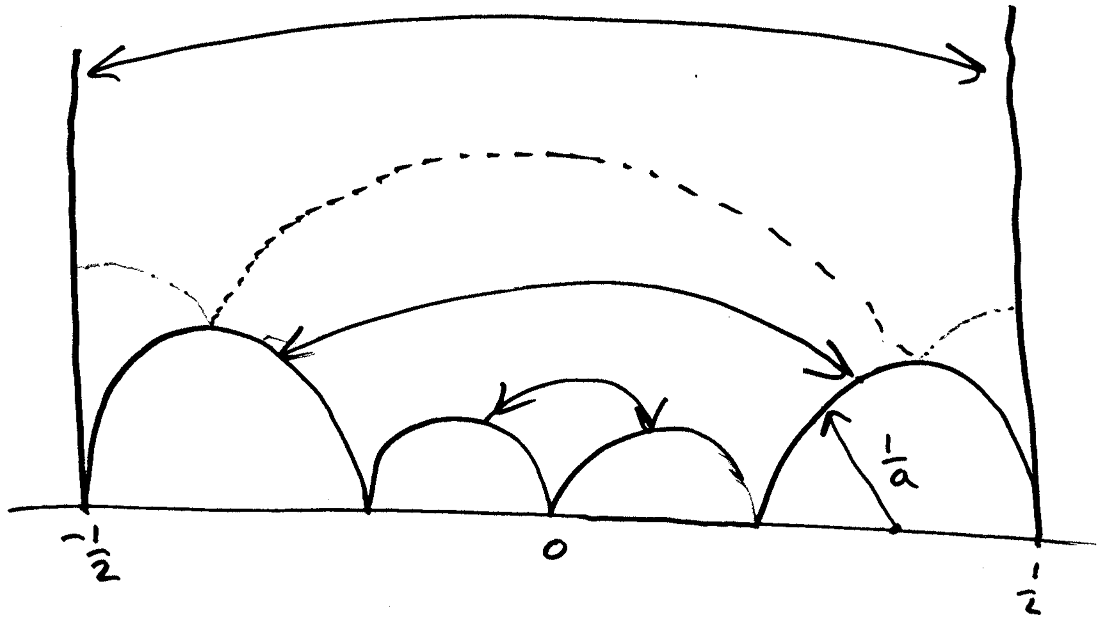
prime geodesics

+ another continuous part + 2 small terms

φ = determinant of scattering matrix Φ

Φ $m \times m$ $\quad m = \#$ of cusps

R_{γ} (9)



$a = 5 \quad \Longleftrightarrow \quad \Gamma_0(5)$

$a = 6 \quad \Longleftrightarrow \quad \Gamma_0(6)$

$a = 8 \quad \Longleftrightarrow \quad \Gamma_0(8)$

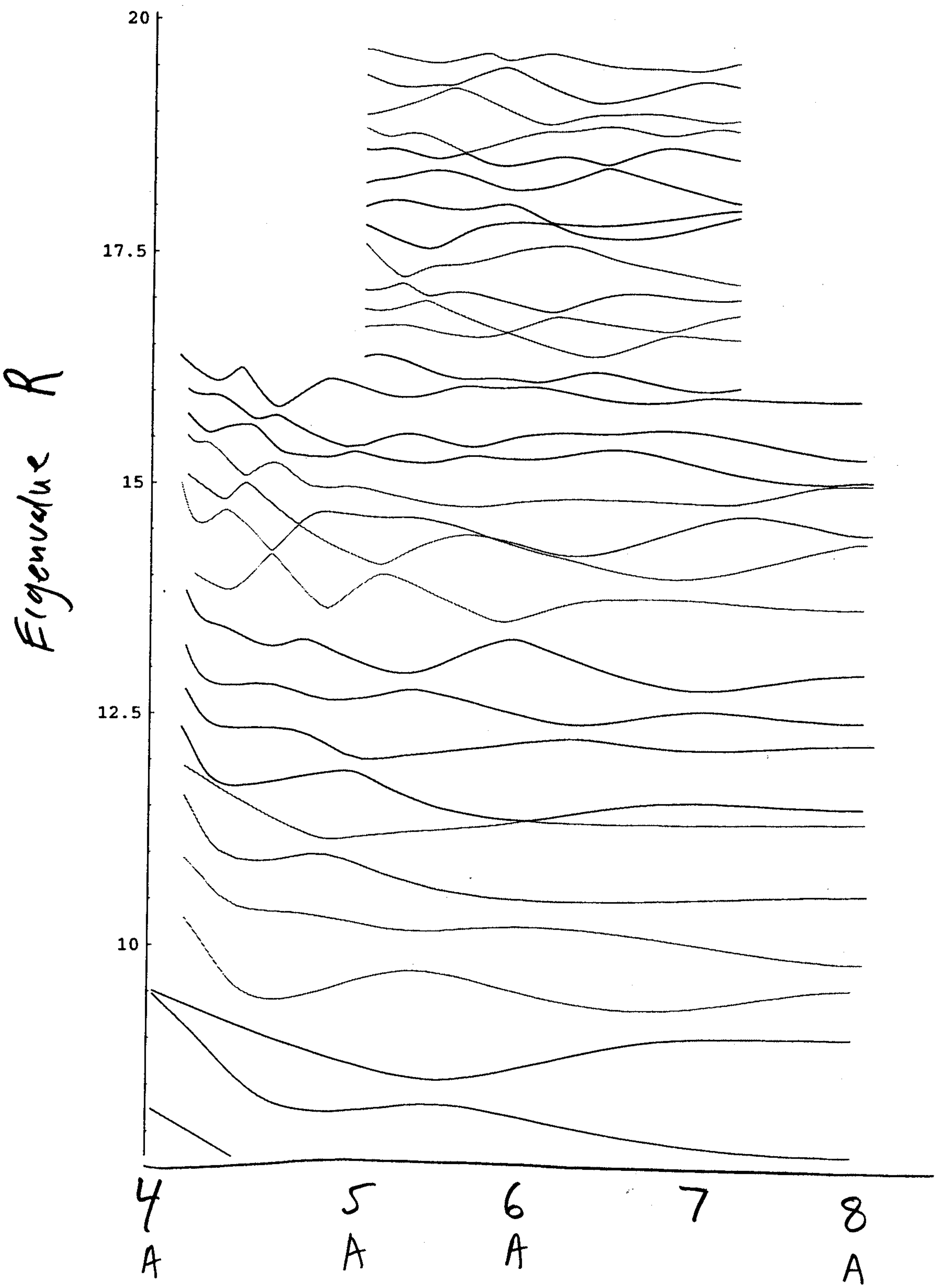
no other arithmetic values

One parameter family of surfaces,
Constant negative curvature, same topology

Expect COE statistics for the eigenvalues

nearest
neighbor
spacing

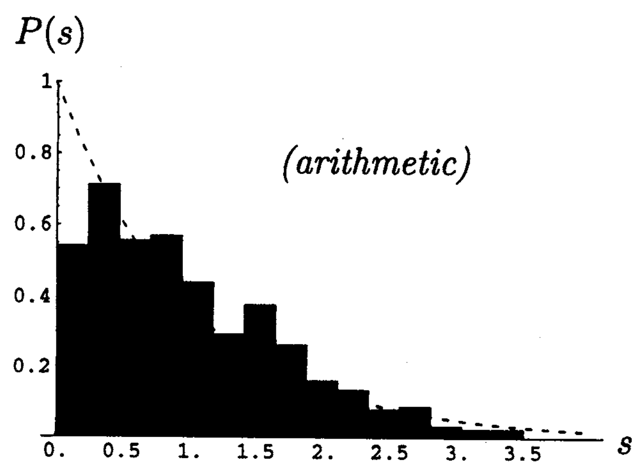
Motion of eigenvalues for $4 \le a \le 8$

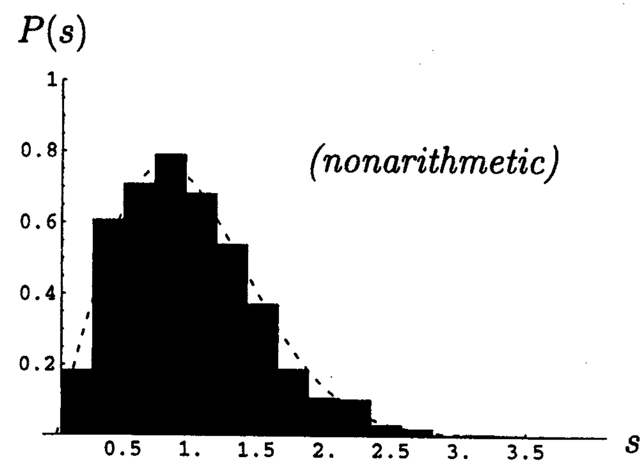


R_y (11)

Level Spacing Distribution

The plots show the normalised level spacings for 250 consecutive energy levels, averaged over the four symmetry types, for the two arithmetic and two nonarithmetic surfaces. The plot on the left is for the arithmetic groups Γ_5 and Γ_6, while the plot on the right is for the non-arithmetic groups $\Gamma_{5.5}$ and $\Gamma_{6.5}$.





The data are consistent with the expectation that in the arithmetic case the spacings should follow a Poisson distribution, while in the non–arithmetic case, the spacings should display a GOE distribution.

5, 6

5.5, 6.5

Ly(12)

$$f(z) = \sqrt{y} \sum_{n=1}^{\infty} a_n K_{iR}(2\pi n y)\, SC(2\pi n x)$$

$$f(z) = \pm f\left(\frac{-1}{Nz}\right) \qquad \text{Fricke} \qquad SC = \sin \text{ or } \cos$$

$$L_f(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}$$

$$\xi_f(s) = \left(\frac{\sqrt{N}}{\pi}\right)^s \Gamma\left(\frac{s+iR+a}{2}\right)\Gamma\left(\frac{s-iR+a}{2}\right) L_f(s)$$

$$= \varepsilon\, \xi_f(1-s)$$

where $a = \begin{cases} 0 & \cos \\ 1 & \sin \end{cases}$

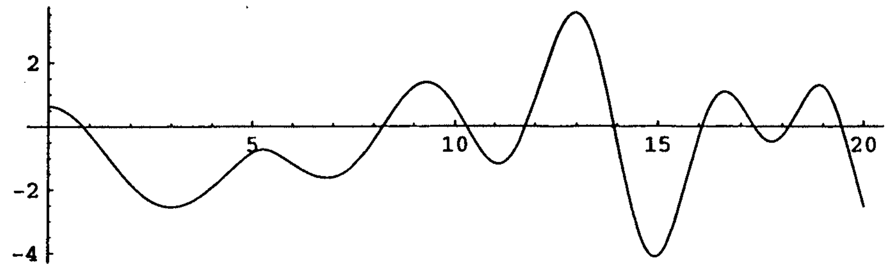
$$\varepsilon = \begin{cases} +1 & \cos + \quad \text{or} \quad \sin - \\ -1 & \cos - \quad \text{or} \quad \sin + \end{cases}$$

R (13)

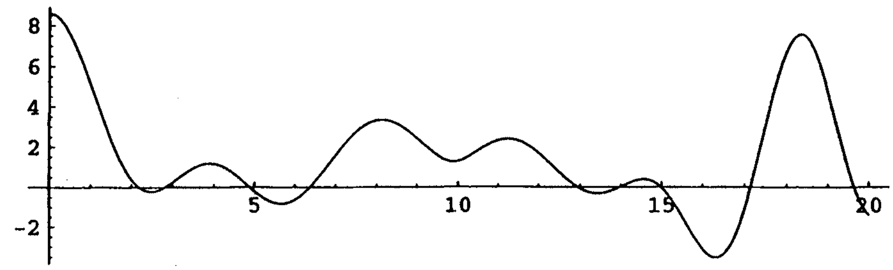
F_L (8)

$\{2, 11, \{\frac{iR}{2}, -\frac{iR}{2}\}, +1\}$
(even +)

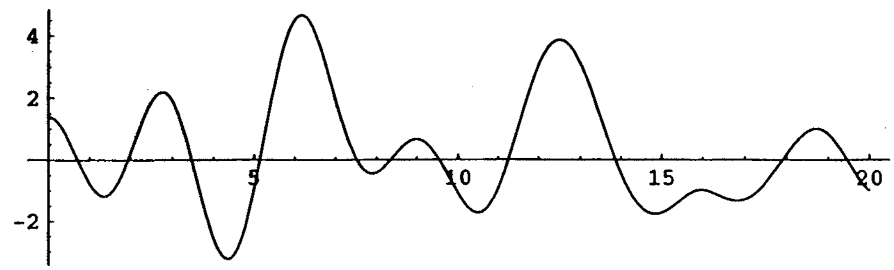
5.26507



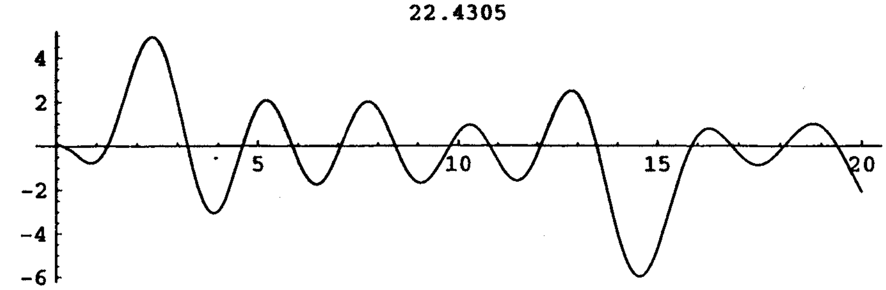
9.86214



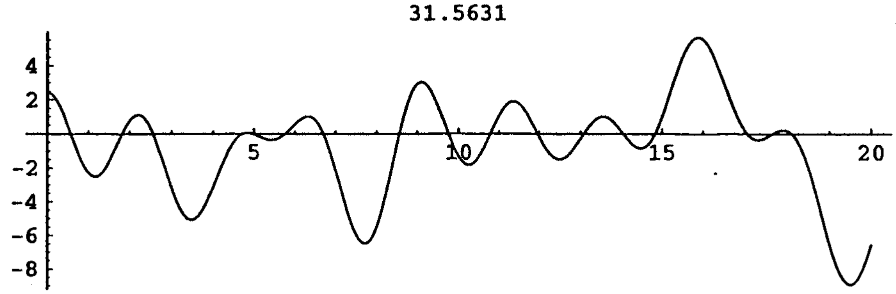
15.9297



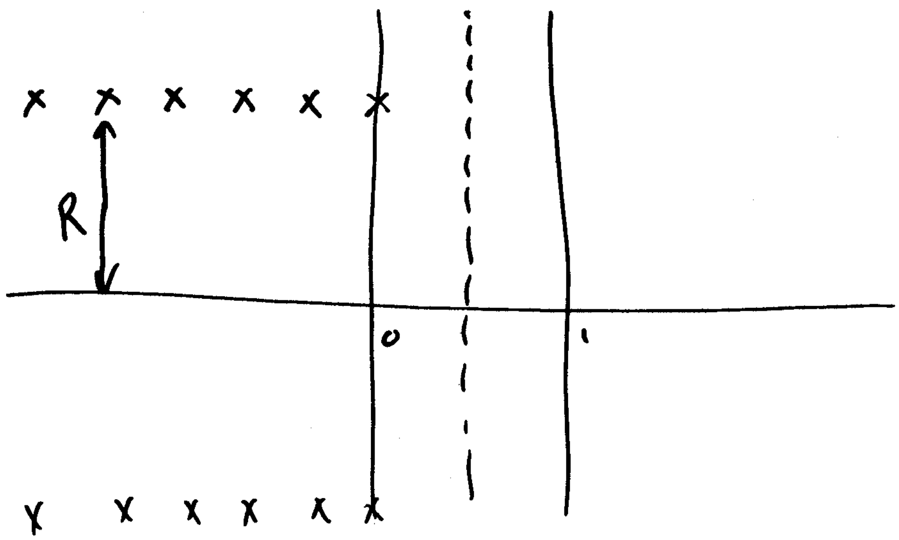
22.4305



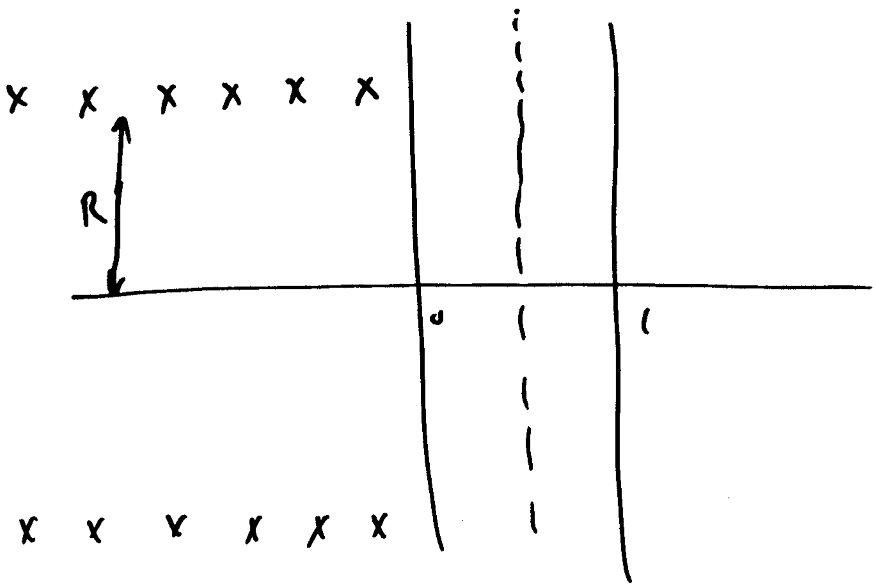
31.5631



Trivial zeros



a=0
(even)



a=1
(odd)

Ry 15
F2 (10/12)