# Families of $L$-functions

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# 1 Introduction

A recent advance in number theory has been the realization of the importance of studying families of $L$-functions and their connections with classical symmetry groups. Often in applications it is sufficient to have control on the average behavior of a family of $L$-functions. Sometimes one can even produce results on individual $L$-functions from average behavior (such as using the amplification method to prove a subonvexity estimate). For these applications it is natural to study the power moments of a family. These moments, besides being useful tools, also contain beautiful structure and are worthy of study in their own right.

In general it is difficult to prove an asymptotic formula for a power moment, in part due to the appearance of subtle off-diagonal terms in the third and fourth moments. Recently, five authors [CFKRS] developed a method for conjecturing a precise formula for any integral moment of a family of $L$-functions. The purpose of this lecture is to describe their method. Essentially everything presented here can be found in greater detail in their paper.

# 2 Families

It seems difficult to come up with a definition of a family of $L$-functions (especially one that would make Bourbaki happy). Nevertheless, it is easy to give specific examples of families and to describe their essential properties.

## 2.1 Example families

*Example (Quadratic Dirichlet characters).* The set of Dirichlet $L$-functions $L(s, \chi_d)$ with $\chi_d$ running over primitive quadratic (real) characters $\pmod d$, $|d| < X$ forms a symplectic family.

*Example (Riemann zeta).* The Riemann zeta function $\{\zeta(\alpha+it) : 0 \leq t \leq T\}$ forms a unitary family in $t$-aspect.

*Example (All Dirichlet characters).* The set $\{L(s, \chi)\}$ of all primitive Dirichlet $L$-functions of conductor $q$ forms a unitary family.

*Example (Hecke L-functions).* The set of Hecke $L$-functions $\{L(s, f)\}$, where $f$ ranges over the set $H_k(q)$ of weight $k$ level $q$ newforms (with either $k$ or $q$ or some combination going to infinity) forms an orthogonal family. Recall

$$L(s, f) = \sum_n \frac{\lambda_f(n)}{n^s},$$

where

$$f(z) = \sum_n \lambda_f(n) n^{(k-1)/2} e(nz) \in H_k(q).$$

## 2.2 The orthogonality relations

The principal property that a family of $L$-functions must have is an orthogonality relation, a concept that we presently describe. Suppose $\mathcal{F}$ is a family of $L$-functions with Dirichlet series $L(s,f) = \sum_n \lambda_f(n) n^{-s}$. Consider the average

$$\delta(m,n) = \lim \frac{1}{|\mathcal{F}|} \sum_{f \in \mathcal{F}} \lambda_f(m) \overline{\lambda_f(n)}. \tag{1}$$

Here the limit must be taken in a sense appropriate to the family; we shall be more explicit when we discuss specific examples. The orthogonality of the family essentially boils down to $\delta(n,n) = 1$ and $\delta(p,1) = 0$ for any prime $p$. Further, $\delta$ must be a multiplicative function; that is, if $(m_1m_2, n_1n_2) = 1$, then $\delta(m_1n_1, m_2n_2) = \delta(m_1,m_2)\delta(n_1,n_2)$. Actually in general we will want to take an average over a product of $l$ coefficients of the $L$-functions for all positive integers $l$, that is we set

$$\delta_l(m_1,\dots,m_k) = \lim \frac{1}{|\mathcal{F}|} \sum_{f \in \mathcal{F}} \lambda_f(m_1) \cdots \lambda_f(m_l) \overline{\lambda_f(m_{l+1}) \dots \lambda_f(m_k)}, \tag{2}$$

and require that $\delta_l$ is multiplicative in the same sense as $\delta(m,n)$.

We now present some examples to illustrate the general picture.

*Example (Riemann Zeta).* Using

$$\frac{1}{T} \int_0^T m^{it} n^{-it} dt = \begin{cases} 1 & \text{if } m = n, \\ \frac{(m/n)^{iT} - 1}{T \log(m/n)} & \text{if } m \neq n, \end{cases}$$

gives

$$\lim_{T \to \infty} \frac{1}{T} \int_0^T m^{it} n^{-it} dt = \begin{cases} 1 & \text{if } m = n, \\ 0 & \text{if } m \neq n. \end{cases}$$

Thus $\delta(m,n)$ is the usual Kronecker delta function. Similarly, the generalizations satisfy $\delta_l(m_1, m_2, \dots, m_k) = \delta(m_1 \dots m_l = m_{l+1} \dots m_k)$.

*Example (Quadratic Dirichlet).* The orthogonality relation is given by

$$\lim_{X \to \infty} \frac{1}{X^*} \sideset{}{^*}\sum_{|d| < X} \chi_d(m) \chi_d(n) = \begin{cases} \prod_{p | mn} (1 + p^{-1})^{-1} & mn = \square, \\ 0 & \text{otherwise}. \end{cases}$$

Here $X^*$ is the number of terms in the sum. The evaluation of the main term is a bit tricky. Note that in this case $\delta(p^2, 1) \neq 0$. Since Dirichlet characters are completely multiplicative we easily see that $\delta(m_1, \dots, m_k) = \delta(m_1 \dots m_k)$ (here we omitted $l$ since it's irrelevant).

*Example (All Dirichlet).* The formula we require is

$$\frac{1}{q^*} \sideset{}{^*}\sum_{\chi \,(\mathrm{mod}\, q)} \chi(m) \overline{\chi}(n) = \frac{1}{q^*} \sum_{\substack{d | (q, m-n) \\ (mn,q) = 1}} \phi(d) \mu(q/d),$$

where $q^* = \sum_{d|q} \phi(d) \mu(q/d)$ is the number of primitive Dirichlet characters modulo $q$. It is important to notice that this sum is not multiplicative in $m$ and $n$, which explains why we need to take a suitable limit of this expression. It turns out to be natural to take a sequence of $q$'s tending to infinity with a fixed set of prime factors (so that the condition $(mn, q) = 1$

remains constant over the sequence). This way of taking the limit forces $d$ to also grow arbitrarily large, because the $\mu(q/d)$ term forces $d$ to remain close to $q$. Consequently $m$ must equal $n$, and the orthogonality relation we get is

$$\delta(m,n) = \begin{cases} 1 & m = n, (mn,q) = 1 \\ 0 & \text{otherwise.} \end{cases} \tag{3}$$

Notice that this would be the orthogonality relation if we summed over all Dirichlet characters (mod $q$), not just primitive characters.

*Example (Hecke).* The Petersson formula gives

$$\sum_{f\in S_k(q)}^{h} \lambda_f(m)\overline{\lambda_f(n)} = \delta_{m,n} + 2\pi i^k \sum_{c\equiv 0 \,(\mathrm{mod}\ q)} \frac{S(m,n;c)}{c} J_{k-1}\left(\frac{4\pi\sqrt{mn}}{c}\right). \tag{4}$$

This is not quite the formula we are looking for, because our family consists of newforms, whereas the Petersson formula has all cusp forms. Nevertheless, by analogy with the Dirichlet $L$-function case, we ignore the issue of primitivity under the supposition that it becomes negligible under the appropriate limit. Using the simple bound $J_{k-1}(x) \ll \min(1, \frac{x}{k})k^{-1/3}$ shows that in the limit as either $k$ or $q$ (or both) tend to infinity, the sum of Kloosterman sums becomes negligible and all that remains is $\delta_{m,n}$.

Iwaniec, Luo, and Sarnak have developed some formulas that express the sum of Fourier coefficients of newforms as a combination of sums of Fourier coefficients of all cusp forms with levels dividing $q$ (see [ILS] Proposition 2.8). Their work lends some support to the above disussion, yet it should be stressed that the passage to the limit in (1) is a heuristic and cannot be mathematically justified.

The orthogonality relation $\delta(m,n) = \delta(m = n)$ for the family of Hecke $L$-functions is the same as that of the Riemann zeta function, but the generalizations $\delta_l(m_1,\ldots,m_k)$ are different. We may reduce the general case to the two-variable case using the Hecke relations

$$\lambda(m_1)\lambda(m_2) = \sum_{\substack{d|(m_1,m_2)\\(d,q)=1}} \lambda(\frac{m_1m_2}{d^2}). \tag{5}$$

The general case can be worked out explicitly using properties of Tchebyshev polynomials; see Lemma 3.1.2 of [CFKRS].

The importance of the multiplicativity of an orthogonality relation is that the Dirichlet series

$$\sum_{m_1,\ldots,m_k} \frac{\delta_l(m_1,\ldots,m_k)}{m_1^{s_1}\ldots m_k^{s_l}}$$

factors into the Euler product

$$\prod_p \sum_{e_1,\ldots,e_k} \frac{\delta_l(p^{e_1},\ldots,p^{e_k})}{p^{e_1s_1+\ldots+e_ks_l}}.$$

The orthogonality of the family means that $\delta_l(p^{e_1},\ldots,p^{e_k}) = 0$ if $\sum e_i = 1$, which has an effect on the locations of poles of the Dirichlet series.

# 3 The recipe for conjecturing moments

In this section we illustrate the general recipe for conjecturing moments by working out a specific example. It is important to understand that this recipe is a kind of mechanical process by which one can arrive at the final answer and that many of the steps can not be justified. In particular, this process can not be used as an outline for a proof of an asymptotic formula because at many steps along the way various subtle main terms are ignored. The amazing thing is that these various errors seem to cancel to give the correct final answer.

## 3.1 Quadratic Dirichlet characters

We begin by writing an approximate functional equation of the form

$$L(s,\chi_d)=\sum_n \frac{\chi_d(n)}{n^s}+X_d(s)\sum_n \frac{\chi_d(n)}{n^{1-s}}+\dots.$$

Here $X_d(s)$ is determined by

(6) $$L(s,\chi_d)=X_d(s)L(1-s,\chi_d),$$

and the dots represent an 'error term.' Actually the Gamma factors implicit in $X_d$ depend on the parity of $\chi_d$ (it is even if $d>0$ and odd if $d<0$) so we treat the two cases separately. The moment of interest is

$$\frac{1}{X^*}\sum_{-d>0} L(\frac{1}{2}+\alpha_1,\chi_d)\dots L(\frac{1}{2}+\alpha_k,\chi_d)g(d/X),$$

where $g$ is a nice test function. Traditionally most authors have studied moments of central values of $L$-functions, but it turns out that the presence of these shifts $\alpha_1,\dots,\alpha_k$ (which we assume are small with respect to $d$ and distinct) reveal hidden structure of the moments in the form of symmetries. They also tend to simplify certain residue calculations because they lead to simple poles. Further, they easily allow for one to produce conjectures for central values of derivatives of a family of $L$-functions.

It turns out that the notation becomes a bit simpler if we consider

$$Z(s,\chi_d):=X_d(s)^{-\frac{1}{2}}L(s,\chi_d),$$

since the approximate functional equation becomes

(7) $$Z(s,\chi_d)=X_d(s)^{-\frac{1}{2}}\sum_n \frac{\chi_d(n)}{n^s}+X_d(s)^{\frac{1}{2}}\sum_n \frac{\chi_d(n)}{n^{1-s}}+\dots$$

The general recipe is as follows.

1. Let

$$Z(s,\alpha_1,\dots,\alpha_k)=Z(s+\alpha_1,\chi_d)\dots Z(s+\alpha_k,\chi_d).$$

2. Replace each $Z$-function with the approximate functional equation (7), ignoring the error term. Multiplying them out gives

$$\text{(product of } X_d \text{ factors)} \sum_{n_1,\ldots,n_k} \text{(summand)}.$$

For general families it is necessary to keep track of the $\epsilon$ factors in the functional equation (e.g. in the case of Hecke $L$-functions the sign in the functional equation would appear here), but for the family of quadratic Dirichlet characters, $\epsilon = 1$.

3. Replace the product of $\epsilon$ factors by its average over the family. (In our case, since $\epsilon = 1$ this step does nothing.)

4. Average the summand over the family (using the orthogonality relation).

5. Extend the range of summation over $n_1, \ldots, n_k$ to all positive integers (in (6) the ranges of summation were deliberately omitted since they would turn out to be irrelevant). Let $M(s, \alpha_1, \ldots, \alpha_k)$ be this sum.

6. The conjecture is

$$\sum_d Z(\frac{1}{2}, \alpha_1, \ldots, \alpha_k) g(d/X) = \sum_d M(\frac{1}{2}, \alpha_1, \ldots, \alpha_k)(1 + O(d^{-\frac{1}{2}}))g(d/X).$$

Now let us work out these steps for our family of quadratic characters. We write

$$Z(\frac{1}{2} + s + \alpha, \chi_d) = \sum_{\varepsilon \in \{-1,1\}} X_d(\frac{1}{2} + \alpha + s)^{-\varepsilon/2} \sum_n \frac{\chi_d(n)}{n^{\frac{1}{2}+\varepsilon(\alpha+s)}}.$$

Thus

$$(8) \quad M(\frac{1}{2} + s, \alpha_1, \ldots, \alpha_k) = \sum_{\varepsilon_1, \ldots \epsilon_k \in \{-1,1\}} X_d(\frac{1}{2} + \alpha_1 + s)^{-\varepsilon_1/2} \ldots X_d(\frac{1}{2} + \alpha_k + s)^{-\varepsilon_k/2}$$
$$\sum_{n_1,\ldots,n_k} \frac{\delta(n_1, \ldots, n_k)}{n_1^{\frac{1}{2}+\varepsilon_1(\alpha_1+s)} \ldots n_k^{\frac{1}{2}+\varepsilon_1(\alpha_k+s)}}.$$

To get a useful expression we need to develop further the sum over $n_1, \ldots, n_k$. Since we will eventually take $s = 0$, we consider the somewhat nicer sum

$$M^*(s) := \sum_{n_1,\ldots,n_k} \frac{\delta(n_1, \ldots, n_k)}{n_1^{s+\varepsilon_1\alpha_1} \ldots n_k^{s+\varepsilon_k\alpha_k}},$$

which has the Euler product

$$M^*(s) = \prod_p \sum_{\substack{e_1,\ldots,e_k \\ e_1+\ldots+e_k \text{ even}}} \frac{\delta(p^{e_1+\ldots+e_k})}{p^{(e_1+\ldots+e_k)s+e_1\varepsilon_1\alpha_1+\ldots+e_k\varepsilon_k\alpha_k}}.$$

Here the $\delta(p^{e_1+\ldots+e_k})$ term is 1 if all the $e_i = 0$ and is $(1 + p^{-1})^{-1}$ otherwise. It is possible to compute this Euler product exactly, but our goal is to extract the analytic behavior of

$M^*(\frac{1}{2})$ in terms of $\alpha_1, \ldots \alpha_k$ (all of which lie in a small neighborhood of the origin). The key point is that only the terms with $e_1 + \ldots + e_k = 2$ can lead to poles. Thus

$$M^*(s) = \prod_p \left(1 + \sum_{1\le i\le j\le k} p^{-2s-\varepsilon_i\alpha_i-\varepsilon_j\alpha_j} + \text{lower order terms}\right)$$
$$= \left(\prod_{1\le i\le j\le k} \zeta(2s + \varepsilon_i\alpha_i + \varepsilon_j\alpha_j)\right) A_k(s),$$

where $A_k(s)$, called the arithmetical factor, is given by an absolutely convergent Dirichlet series for $\mathrm{Re}(s) > \frac{1}{2} - \delta$ for some $\delta > 0$ and all $\alpha_i$ sufficiently close to the origin.

Thus

$$M(\frac{1}{2}, \alpha_1, \ldots, \alpha_k) = \sum_{\varepsilon_1,\ldots\epsilon_k\in\{-1,1\}} X_d(\frac{1}{2}+\alpha_1)^{-\varepsilon_1/2} \ldots X_d(\frac{1}{2}+\alpha_k)^{-\varepsilon_k/2}$$
$$A_k \prod_{1\le i\le j\le k} \zeta(1 + \varepsilon_i\alpha_i + \varepsilon_j\alpha_j),$$

where $A_k = A_k(\frac{1}{2})$. This is a curious expression for the $k$-th moment because the various factors $\zeta(1 + \varepsilon_i\alpha_i + \varepsilon_j\alpha_j)$ are not holomorphic in a neighborhood of the origin, whereas obviously the $k$-th moment is holomorphic there. It requires extra work to show that the sums over the $\varepsilon_j$ cause these poles to all cancel. This permutation sum can be written compactly as a contour integral using Lemma 2.5.2 of [CFKRS], which we reproduce here.

**Lemma 3.1.** *Suppose $F$ is a symmetric function of $k$ variables, regular near $(0, \ldots, 0)$, and $f(s)$ has a simple pole of residue 1 at $s = 0$ and is otherwise analytic in a neighborhood of $s = 0$, and let*

$$K(a_1, \ldots, a_k) = F(a_1, \ldots, a_k) \prod_{1\le i\le j\le k} f(a_i + a_j).$$

*If $\alpha_i + \alpha_j$ are contained in the region of analyticity of $f(s)$ then*

$$\sum_{\varepsilon_1,\ldots,\varepsilon_k\in\{-1,1\}} K(\varepsilon_1\alpha_1, \ldots, \varepsilon_k\alpha_k) =$$
$$\frac{(-1)^{k(k-1)/2}}{(2\pi i)^k}\frac{2^k}{k!} \int \cdots \int K(z_1, \ldots, z_k) \frac{\Delta(z_1^2, \ldots, z_k^2)^2 \prod_{j=1}^k z_j}{\prod_{1\le i,j\le k}(z_i - \alpha_j)(z_i + \alpha_j)} dz_1 \ldots dz_k,$$

*where the paths of integration are closed contours enclosing the $\pm\alpha_j$, and where $\Delta$ is the Vandermonde:*

$$\Delta(z_1, \ldots, z_k) = \prod_{1\le i<j\le k} (z_j - z_i).$$

This Lemma immediately implies that $M(\frac{1}{2}, \alpha_1, \ldots, \alpha_k)$ is holomorphic near the origin.

## 3.2 Illegality

It is important to understand how the recipe differs from reality.

1. The first step is a definition.

2. The use of an approximate functional equation has been a popular way for many researchers to study a moment. For example one could write

$$L(s,\chi_d) = \sum_{n=1}^{\infty} \frac{\chi_d(n)}{n^s} V_s(n/\sqrt{|d|}) + X_d(s) \sum_{n=1}^{\infty} \frac{\chi_d(n)}{n^{1-s}} V_{1-s}(n/\sqrt{|d|}),$$

   where $V_s(x)$ is a smooth function (there are many choices for $V$). Due to the decay of $V$, these infinite sums over $n$ can be truncated to $n \le |d|^{\frac{1}{2}+\varepsilon}$ with a very small error.

3. Of course $\epsilon$ is not independent of the coefficients of the $L$-functions so this is a heuristic. Yet, it seems to be not far from the truth.

4. The recipe commits a felony here by completely ignoring off-diagonal terms. Here the issue is that in the derivation of the orthogonality relation, $n_1, \dots, n_k$ were fixed and the conductor became arbitrarily large. In practice, the approximate functional equation allows each $n_i$ to be as large as the square root of the conductor. Main terms get ignored in this step.

5. Another illegal operation is performed in this stage by allowing $n_1, \dots, n_k$ to run over all positive integers. As was already mentioned, the sizes of $n_i$ and the conductor $d$ are inextricably linked. This step again ignores this connection.

6. In our above computation of $M(\frac{1}{2}, \alpha_1, \dots, \alpha_k)$ there was actually a subtlety that was hidden, namely that (8) does not converge anywhere. The meromorphic continuation was obtained by studying the sum over $n_1, \dots n_k$ for each choice of $\epsilon_1, \dots, \epsilon_k$, and finding the meromorphic continuation for each such sum.

The key differences between the recipe and reality are in steps 4 and 5.

It is remarkable that such a simple mechanism (which completely ignores the complicated way that the variables $n_1, \dots, n_k$ relate to the conductor) can seem to give the correct conjecture on the moments. Perhaps the truly amazing thing is that the correct (finite) orthogonality relation of a family of $L$-functions approximates the limited orthogonality relation.

## 3.3 An elaboration of the moment conjecture

A refined calculation of the arithmetical factor $A_k$ and an application of Lemma 3.1 gives

**Conjecture.** *Let $X_d(s) = |d|^{\frac{1}{2}-s} X(s)$, where*

$$X(s) = \pi^{s-\frac{1}{2}} \frac{\Gamma(\frac{2-s}{2})}{\Gamma(\frac{1+s}{2})}.$$

*The conjecture is*

$$\sum_d{}^{*} L(\frac{1}{2}, \chi_d)^k g(|d|) = \sum_d{}^{*} Q_k(\log |d|)(1 + O(|d|^{-\frac{1}{2}+\varepsilon}) g(|d|),$$

*where $Q_k$ is the polynomial of degree $\frac{1}{2}k(k+1)$ given by*

$$Q_k(x) = \frac{(-1)^{k(k-1)/2}2^k}{k!} \frac{1}{(2\pi i)^k} \int \dots \int G(z_1, \dots, z_k) \frac{\Delta(z_1^2, \dots, z_k^2)^2}{\prod_{j=1}^k z_j^{2k-1}} e^{\frac{x}{2}(\sum_{j=1}^k z_j)} dz_1 \dots dz_k,$$

*where*

$$G(z_1, \dots, z_k) = A_k(z_1, \dots, z_k) \left( \prod_{j=1}^k X(\frac{1}{2} + z_j)^{-\frac{1}{2}} \right) \left( \prod_{1 \le i \le j \le k} \zeta(1 + z_i + z_j) \right),$$

*and $A_k$ is the Euler product defined by*

$$A_k(z_1, \dots, z_k) = \prod_p \prod_{1 \le i \le j \le k} \left(1 - p^{-1-z_i-z_j}\right)$$
$$\times \left( \frac{1}{2} \left( \prod_{j=1}^k (1 - p^{-\frac{1}{2}-z_j})^{-1} + \prod_{j=1}^k (1 + p^{-\frac{1}{2}-z_j})^{-1} \right) + p^{-1} \right) (1 + p^{-1})^{-1}.$$

This is Conjecture 1.5.3 of [CFKRS].

# References


[CFKRS] J. B. Conrey, D. W. Farmer, J. P. Keating, M. O. Rubinstein, N. C. Snaith, *Integral moments of L-functions*. Proc. London Math. Soc. (3) 91 (2005), no. 1, 33–104.

[ILS] H. Iwaniec, W. Luo and P. Sarnak, *Low lying zeros of families of L-functions*, Inst. Hautes Etudes Sci. Publ. Math. no. 91, 55-131 (2001).