Conrey catch-up

$$\xi(s) = \xi(1-s)$$
$$\|$$
$$G(s)L(s)$$

$$\left[\xi(s) = \sum_{n \le X} \frac{1}{n^s} + \sum_{n \le Y} \frac{1}{n^{1-s}} \chi(s) + O(X^{-\sigma} + t^{\sigma - 1/2} Y^{1-\sigma}) \qquad XY \le \frac{t}{2\pi} \right]$$

$$g(-s) = g(s)$$
$$g(0) = 1$$

$$\frac{1}{2\pi i} \int_{1-i\infty}^{1+i\infty} \xi(s+1/2)\, g(s) \frac{ds}{s} = \frac{1}{2\pi i} \int_{1-i\infty}^{1+i\infty} \sum_{n=1}^{\infty} \frac{a_n}{n^{s+1/2}} G(s+\tfrac{1}{2})\, g(s) \frac{ds}{s}$$

$$= \sum_{n=1}^{\infty} \frac{a_n}{\sqrt{n}} W(n)$$

$$W(x) = \frac{1}{2\pi i} \int_{1-i\infty}^{1+i\infty} G(s+1/2)\, g(s)\, x^{-s} \frac{ds}{s}$$

Move the contour of integration to (-1)

Residue @ $s=0$

$$\frac{1}{2\pi i} \int_{-1+i\infty}^{-1+i\infty} \xi(s+1/2)\, g(s) \frac{ds}{s} + \xi(1/2)$$
$$\|$$
$$-\frac{1}{2\pi i} \int_{1-i\infty}^{1+i\infty} \xi(\tfrac{1}{2}-s)\, g(s) \frac{ds}{s} + \xi(1/2) \qquad \text{via } -s \mapsto -s$$
$$\|$$
$$-\frac{1}{2\pi i} \int_{1-i\infty}^{1+i\infty} \xi(\tfrac{1}{2}+s)\, g(s) \frac{ds}{s} + \xi(1/2) \qquad \text{by functional equation.}$$

So we got

$$\xi(1/2) = 2\sum_{n=1}^{\infty} \frac{a_n W(n)}{\sqrt{n}}$$

$$\|$$

$$G(\tfrac{1}{2}) L(1/2)$$

$$L(1/2) = 2\sum_{n=1}^{\infty} \frac{a_n W(n)}{\sqrt{n}\, G(1/2)}$$

$$\frac{W(x)}{G(1/2)} = \frac{1}{2\pi i}\int_{1-i\infty}^{1+i\infty} \frac{G(s+1/2)}{G(1/2)} g(s)\, x^{-s} \frac{ds}{s}$$

We'd like to know that $\frac{W(x)}{G(1/2)} = \begin{cases} 1 + O_G(x^{-A}) & \text{if } x < 1 \\ O_G(x^{-A}) & \text{if } x > 1 \end{cases}$

where A is a number s.t. $G(s+1/2)g(s)$ is analytic for $|\mathrm{Re}\, s| < A+1$

<u>Example</u>

$$L(s,\chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s} \quad ; \quad \chi \text{ primitive, even, mod } q$$

$$\xi(s,\chi) = \left(\frac{\pi}{q}\right)^{-s/2} \Gamma\left(\frac{s}{2}\right) L(s,\chi) = \varepsilon_\chi\, \xi(1-s,\overline{\chi})$$

$$\xi(s) = \xi(s,\chi)\xi(s,\overline{\chi}) = \xi(1-s)$$

$$\xi(s) = \left(\frac{\pi}{q}\right)^{-s} \Gamma\left(\frac{s}{2}\right)^2 L(s,\chi) L(s,\overline{\chi})$$

$$\frac{1}{2\pi i}\int_{1-i\infty}^{1+i\infty} \xi(s+\tfrac{1}{2})\, g(s)\,\frac{ds}{s} = \frac{1}{2\pi i}\int_{1-i\infty}^{1+i\infty} \Gamma\left(\frac{s}{2}+\frac{1}{4}\right)^2 \left(\frac{\pi}{q}\right)^{-s-\frac{1}{2}} L\left(\tfrac{1}{2}+s,\chi\right)$$

$$\times\, L\left(\tfrac{1}{2}+\overline{\chi}\right) g(s)\,\frac{ds}{s}$$

$$= \left(\frac{\pi}{q}\right)^{-1/2} \sum_{m,n} \frac{\chi(m)\overline{\chi(n)}}{\sqrt{mn}}\; \underbrace{\frac{1}{2\pi i}\int_{1-i\infty}^{1+i\infty} \Gamma\left(\frac{s}{2}+\frac{1}{4}\right)^2 g(s)\left(\frac{\pi mn}{q}\right)^{-s}\frac{ds}{s}}_{\stackrel{!!}{} \; W\left(\frac{\pi mn}{q}\right)\Gamma\left(\frac{1}{4}\right)^2}$$

$$\text{LHS} = \xi\left(\tfrac{1}{2}\right) + \int_{-1-i\infty}^{-1+i\infty} \cdots$$

$$= \left(\frac{\pi}{q}\right)^{-1/2} \Gamma\left(\frac{1}{4}\right)^2 \left|L\left(\tfrac{1}{2},\chi\right)\right|^2 + \cdots$$

$$\left|L\left(\tfrac{1}{2},\chi\right)\right|^2 = 2\sum_{m,n=1}^{\infty} \frac{\chi(m)\overline{\chi}(n)}{\sqrt{mn}}\, W\left(\frac{\pi mn}{q}\right)$$

Where $W(x) = \dfrac{1}{2\pi i}\displaystyle\int_{1-i\infty}^{1+i\infty} \frac{\Gamma\left(\frac{s}{2}+\frac{1}{4}\right)^2}{\Gamma\left(\frac{1}{4}\right)^2}\, g(s)\, x^{-s}\,\frac{ds}{s}$

By good choice of $g(s)$, $W(x) \ll \dfrac{1}{x^a}$

If $\gcd(mn,q)=1$ then

$$\sum_{\chi \bmod q}^{*} \chi(m)\overline{\chi}(n) = \sum_{\substack{d\mid q\\ d\mid (m-n)}} \mu\left(\frac{q}{d}\right)\phi(d)$$

$$\frac{1+\chi(-1)}{2} = \begin{cases} 1 & \text{if } \chi \text{ even}, \\ 0 & \text{if } \chi \text{ odd}. \end{cases}$$

$$\sum_{\chi \bmod q}^{*} \left(\frac{1+\chi(-1)}{2}\right) \chi(m)\chi(n) = \sum_{\substack{d|q \\ d|m-n}} \cdots + \sum_{\substack{d|q \\ d|m+n}} \cdots$$

# of even chars

$$\sum_{\substack{\chi \bmod q \\ \text{even}}}^{*} |L(\tfrac{1}{2},\chi)|^2 = \varphi^{\text{even}}(q) \, 2 \sum_{\gcd(m,q)=1} \frac{W\left(\frac{\pi m^2}{q}\right)}{m} + \sum_{m \neq n} \text{off-diagonal terms}$$

$$\text{Diagonal term} = \sum_{\gcd(m,q)=1} \frac{1}{m} \frac{1}{2\pi i} \int_{1-i\infty}^{1+i\infty} \frac{\Gamma\left(\frac{s}{2}+\frac{1}{4}\right)^2}{\Gamma\left(\frac{1}{4}\right)^2} g(s) \left(\frac{\pi m^2}{q}\right)^{-s} \frac{ds}{s}$$

$$= \frac{1}{2\pi i} \int_{1-i\infty}^{1+i\infty} \frac{\Gamma\left(\frac{s}{2}+\frac{1}{4}\right)^2}{\Gamma(1/4)^2} g(s) \left(\frac{\pi}{q}\right)^{-s} \zeta(1+2s) \prod_{p|q} \left(1 - \frac{1}{p^{1+2s}}\right) \frac{ds}{s}.$$

$$\left[\sum_{\gcd(m,q)=1} \frac{1}{m^{1+2s}} = \zeta(1+2s) \prod_{p|q} \left(1 - \frac{1}{p^{1+2s}}\right) \right]$$

2 poles @ $s=0$

Bound off-diagonals

$$\sum_{\substack{mn \leq X \\ m \equiv n \bmod d}} \frac{1}{\sqrt{mn}} + \sum_{mn > X} \frac{W\left(\frac{\pi mn}{q}\right)}{\sqrt{mn}}$$

Use bounds for W and choose X optimally to get bound $\frac{q^{1/2+\varepsilon}}{d}$