Conrey 3

Motivation

$s = \frac{1}{2} + it$

$$\int_0^T \zeta(s+\alpha)\zeta(s+\beta)\zeta(1-s+\gamma)\zeta(1-s+\delta)\,dt$$

$$= \int_0^T Z(\alpha,\beta;\gamma,\delta) + \left(\frac{t}{2\pi}\right)^{-\alpha-\gamma} Z(-\gamma,\beta;-\alpha,\delta)$$

$$+ \left(\frac{t}{2\pi}\right)^{-\alpha-\delta} Z(-\delta,\beta;\gamma,-\alpha) + \left(\frac{t}{2\pi}\right)^{-\beta-\gamma} Z(\alpha,-\gamma;-\beta,\delta)$$

$$+ \left(\frac{t}{2\pi}\right)^{-\beta-\delta} Z(\alpha,-\delta;\gamma,-\beta) + \left(\frac{t}{2\pi}\right)^{-\alpha-\beta-\gamma-\delta} Z(-\gamma,-\delta;-\alpha,-\beta)$$

$$\times\, dt + O(T^{2/3+\varepsilon})$$

$$Z(\alpha,\beta;\gamma,\delta) = \frac{\zeta(1+\alpha+\gamma)\,\zeta(1+\alpha+\delta)\,\zeta(1+\beta+\gamma)\,\zeta(1+\beta+\delta)}{\zeta(2+\alpha+\beta+\gamma+\delta)}$$

Random matrix theory

Let $\Lambda_A(s) = \prod_{n=1}^{N} (1 - s e^{-i\theta_n})$

$$\int_{U(N)} \Lambda_A(e^{-\alpha})\Lambda_A(e^{-\beta})\Lambda_{A^*}(e^{-\gamma})\Lambda_{A^*}(e^{-\delta})\,dA_N$$

$$= \mathcal{Z}(\alpha,\beta,\gamma,\delta) + e^{-(\alpha+\gamma)N}\mathcal{Z}(-\gamma,\beta;-\alpha,\delta) + \cdots$$

Conrey, Forrester, Snaith

(Baik, Deift, Strahov)

Heine's identity

$$W(x) \text{ weight}, \qquad I \subseteq \mathbb{R}$$

$$d\mu(x) = W(x)\,dx$$

Monic orthogonal polynomials $\pi(N,x) = N^{th}$ polynomial, orthogonal w.r.t weight W.

$$P_N(x_1, \dots, x_N) = \frac{\Delta(x)^2}{N!} \prod_{n=1}^{N} W(x_n)$$

Let $$D(X,Y) = \prod_{\substack{x \in X \\ y \in Y}} (y - x)$$

$$\int_{I^N} D(X, \{y\}) \frac{\Delta(x)^2}{N!} \prod_{n=1}^{N} W(x_n)\, dx_n \qquad \text{(Heine)}$$

$$= \pi(N,y) \int_{I^N} \frac{\Delta(x)^2}{N!} \prod_{n=1}^{N} W(x_n)\, dx_n$$

[Want $$\int_{I^N} \frac{D(X,A_K)}{D(X,C_Q)} \frac{\Delta(x)^2}{N!} \prod_{n=1}^{N} W(x_n)\, dx_n$$]

$$\int_I y^r W(y) \int_{I^N} D(X, \{y\}) \frac{\Delta(x)^2}{N!} \prod_{n=1}^{N} W(x_n) dx_n$$

- $\Delta(x)^2 \longrightarrow N!\, \Delta(x) \prod_{n=1}^{N} x_n^{n-1}$ (Exercise ?)

[<u>Hint</u>: $\Delta(x) = \sum_{\sigma \in S_N} \mathrm{sgn}(\sigma) \prod_{n=1}^{N} x_{\sigma(n)}^{n-1}$]

$$= \int_I y^r W(y) \int_{I_N} D(X, \{y\}) \Delta(x) \prod_{n=1}^{N} W(x_n) x_n^{n-1} dx_n \, dy$$

put $y = x_{N+1}$

$$= \int_{I^{N+1}} x_{N+1}^{r-N} \Delta(x_1, \dots, x_{N+1}) \prod_{n=1}^{N+1} W(x_n) x_n^{n-1} dx_n$$

$$\det \begin{vmatrix} 1 & 1 & 1 & \cdots & 1 \\ x_1 & x_2 & & \cdots & x_{N+1} \\ x_1^2 & x_2^2 & & \cdots & x_{N+1}^2 \\ \vdots & \vdots & & \ddots & \vdots \end{vmatrix} = \det \begin{vmatrix} 1 & 1 & \cdots \\ \pi(1, x_1) & \pi(1, x_2) & \cdots \\ \pi(2, x_1) & \pi(2, x_2) & \cdots \\ \vdots & \vdots & \ddots \end{vmatrix}$$

$$= \det \left(\int \pi_{m-1}(x_n) W(x_n) x_{n-}^{n-1} dx_n \right)$$

$$\times \underbrace{\int \pi(x_{N+1}) x_{N+1}^{r} \, dx_{N+1}}_{\text{is } 0 \text{ if } r<N.}$$

<u>Exercise</u> Finish this off.

<u>Hint</u> $$\frac{1}{\prod_{n=1}^{N} (y - x_n)} = \sum_{n=1}^{N} \frac{1}{y - x_n} \prod_{m=1}^{N} \frac{1}{(x_n - x_m)}$$