B. Winn

Conrey 2

Selberg Class

Ramanujan bound

$$- \ L(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}, \qquad a_n \ll n^{\varepsilon}, \quad a_1 = 1$$

$f(z)$ is of order at most r if $|f(z)| \leq e^{|z|^{r+\varepsilon}}$

4 Axioms: - $(s-1)^m L(s)$ an entire function of order ≤ 1 for some $m \in \mathbb{Z}$

$\left(\overline{f}(z) := \overline{f(\overline{z})}\right)$

$$- \ Q^s \prod_{j=1}^{J} \Gamma(\omega_j s + \mu_j) L(s) = \Phi(s) = \varepsilon \overline{\Phi}(1-s)$$

$$Q > 0, \quad \omega_j > 0, \quad \mathrm{Re}\,\mu_i \geq 0, \quad \varepsilon \in \mathbb{C}$$

- Euler product

$$\log L(s) = \sum_{n=1}^{\infty} \frac{b_n}{n^s}; \quad b_n = 0 \text{ unless } n = p^k$$

$$\exists\, \theta < \tfrac{1}{2} \text{ s.t. } b_n \ll n^{\theta}.$$

Degree $d = 2 \sum_{j=1}^{J} \omega_j$. Conjecture $d \in \mathbb{Z}$

Degree 1: there is $\zeta(s)$, $L(s+i\alpha;\chi)$ χ primitive Dirichlet character, $\alpha \in \mathbb{R}$.

$$q^{s/2} \prod_{j=1}^{d} \pi^{-s/2} \Gamma\left(\frac{s}{2} + \mu_j\right) L(s) = \Phi(s) = \varepsilon \overline{\Phi}(1-s)$$

Is q an integer? Can we write $L(s) = \prod \prod^{d} \left(1 - \frac{\alpha_{j,p}}{p^s}\right)^{-1}$?

- Siegel modular forms.
- Higher symmetric powers.

Averages

$S_k(\Gamma_0(q))$ - Hecke basis $H_k(q)$

$$\sum_{f \in H_k(q)} L_f(1/2)^2 \quad ; \quad L_f(s) = \sum_{n=1}^{\infty} \frac{\lambda_f(n)}{n^s}$$

$\boxed{k=2, \ q = \text{prime}}$ everything is primitive and newform.

$$\left(\frac{2\pi}{\sqrt{q}}\right)^{-s} \Gamma\left(s + \frac{k-1}{2}\right) L_f(s) = \xi_f(s) = \pm \xi_f(1-s)$$

Consider $+$

$$L_f\left(\tfrac{1}{2}\right)^2 = 2 \sum_{m,n=1}^{\infty} \frac{\lambda_f(m)\lambda_f(n)}{\sqrt{mn}} W\left(\frac{4\pi^2 mn}{q}\right)$$

an approximate identity.

$$\sum_{f \in H_k(q)}^{h} L_f\left(\tfrac{1}{2}\right)^2 = 2 \sum_{m,m} \frac{1}{\sqrt{mn}} W\left(\frac{4\pi^2 mn}{q}\right) \sum_{f \in H_k(q)}^{h} \lambda_f(m)\lambda_f(n).$$

$$= 2 \sum_{m,n} \frac{1}{\sqrt{mn}} W\left(\frac{4\pi^2 mn}{q}\right) \left(\delta_m^n + \frac{1}{(2\pi i)^k} \sum_{c \equiv 0 (q)} \frac{S(m,n,c)}{c} J_{k-1}\left(\frac{4\pi\sqrt{mn}}{c}\right)\right)$$

(Petersson formula)

$$(\mathrm{I}) \leq \sum_{d \mid A} d^{1/2} \sum_{\substack{r \leq X \\ r \equiv 0\ (d)}} \frac{\tau(r)}{\sqrt{r}}$$

$$\ll \sum_{d \mid A} d^{1/2} \sum_{r \leq X/d} \frac{\tau(rd)}{\sqrt{rd}}$$

$$\ll \sum_{d \mid A} \tau(d) \sum_{r \leq \frac{X}{d}} \frac{\tau(r)}{\sqrt{r}}$$

$\ll$ about $\sqrt{X}$

(II) gives something similar. Collecting together we get

$$\frac{\tau(q)}{\sqrt{q}} \gcd(m,n,q)^{1/2} \frac{(mn)^{1/4}}{\sqrt{q}} \ll (mn)^{1/4} \frac{\gcd(m,n,q)^{1/2} q^{\varepsilon}}{q}$$

$$\frac{1}{q} \sum_{mn \ll q} \frac{1}{\sqrt{mn}} (mn)^{1/4} \gcd(m,n,q)^{1/2} q^{\varepsilon} \ll q^{-1/4+\varepsilon}.$$

due to weight

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- $(s-1)^m L(s)$ an entire function of order ≤ 1 for some $m \in \mathbb{Z}$
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Is $|\alpha_{j,p}| \in \{0,1\}$?

Degree 2: Should be L-functions associated with :-

- Primitive holomorphic cusp forms
- Primitive Maaß cusp forms.

χ is a character mod N

$$f \in S_k(\Gamma_0(N), \chi)$$

$$f\left(\frac{az+b}{cz+d}\right) = \chi(d)(cz+d)^k f(z)$$

Degree 3: Symmetric square of degree 2

$$L(s) = \prod_p \left(1 - \frac{\alpha_p}{p^s}\right)^{-1}\left(1 - \frac{\alpha_p'}{p^s}\right)^{-1}$$

$$L(\mathrm{sym}^2, s) = \prod_p \left(1 - \frac{\alpha_p^2}{p^s}\right)^{-1}\left(1 - \frac{\alpha_p \alpha_p'}{p^s}\right)^{-1}\left(1 - \frac{(\alpha_p')^2}{p^s}\right)^{-1}$$

Are there other degree 3 things ?

Degree 4:

$$L_f(s) = \prod_p \left(1 - \frac{\alpha_p}{p^s}\right)^{-1}\left(1 - \frac{\alpha_p'}{p^s}\right)^{-1}$$

$$L_g(s) = \prod_p \left(1 - \frac{\beta_p}{p^s}\right)^{-1}\left(1 - \frac{\beta_p'}{p^s}\right)^{-1}$$

- $$L_{f\times g}(s) = \prod_p \left(1 - \frac{\alpha_p\beta_p}{p^s}\right)^{-1}\left(1 - \frac{\alpha_p\beta_p'}{p^s}\right)^{-1}\left(1 - \frac{\alpha_p'\beta_p}{p^s}\right)^{-1}\left(1 - \frac{\alpha_p'\beta_p'}{p^s}\right)$$

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$$= 2\sum_{m,n} \frac{1}{\sqrt{mn}} W\left(\frac{4\pi^2 mn}{q}\right)\Bigg(\delta_m^n + \frac{1}{(2\pi i)^k} \sum_{c\equiv 0(q)} \frac{S(m,n,c)}{c} J_{k-1}\left(\frac{4\pi\sqrt{mn}}{c}\right)\Bigg)$$

(Petersson formula)

Off-diagonal

$$S(m,n,c) = \sum_{x \bmod c}{}' e\left(\frac{mx + n\bar{x}}{c}\right) \qquad x\bar{x} \equiv 1 \pmod c$$

Weil: $$|S(m,n,c)| \leqslant (\gcd(m,n,c))^{1/2} \tau(c)\sqrt{c}$$

divisor function

Bessel: $$J_{k-1}(x) \ll \min\left(1, \frac{x}{k}\right)$$

Bound $$\sum_{c \equiv 0} \frac{S(m,n,c)}{c} J_{k-1}\left(\frac{4\pi\sqrt{mn}}{c}\right)$$

Let $c = qr$

$$= \sum_{r=1}^{\infty} \frac{S(m,n,qr)}{qr} J_{k-1}\left(\frac{4\pi\sqrt{mn}}{qr}\right)$$

$$\ll \frac{1}{q}\sum_{r=1}^{\infty} \frac{\gcd(m,n,rq)^{1/2}\tau(rq)\sqrt{rq}}{r} \min\left(1, \frac{4\pi\sqrt{mn}}{qkr}\right)$$

$$\ll \frac{\tau(q)}{\sqrt{q}} \gcd(m,n,q)^{1/2} \sum_{r=1}^{\infty} \frac{\gcd(m,n,r)^{1/2}}{\sqrt{r}} \tau(r) \times \min\left(1, \frac{4\pi\sqrt{mn}}{qkr}\right)$$

$A = \gcd(m,n)$

$X = \dfrac{4\pi\sqrt{mn}}{qk}$

$$\sum_{r=1}^{\infty} \frac{\tau(r)\gcd(A,r)^{1/2}}{\sqrt{r}} \min\left(1, \frac{X}{r}\right)$$

$$= \underbrace{\sum_{r \leqslant X} \frac{\tau(r)\gcd(A,r)^{1/2}}{\sqrt{r}}}_{(\mathrm{I})} + X \underbrace{\sum_{r > X} \frac{\tau(r)\gcd(A,r)^{1/2}}{r^{3/2}}}_{(\mathrm{II})}$$

Now, $\gcd(A,r)^{1/2} \leqslant \sum_{d \mid A} d^{1/2}$

$$(\mathrm{I}) \leq \sum_{d|A} d^{1/2} \sum_{\substack{r \leq X \\ r \equiv 0\ (d)}} \frac{\tau(r)}{\sqrt{r}}$$

$$\ll \sum_{d|A} d^{1/2} \sum_{r \leq X/d} \frac{\tau(rd)}{\sqrt{rd}}$$

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