B. Winn



# Conrey

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \qquad s = \sigma + it, \qquad \sigma > 1$$

$$= \prod_p \left(1 - \frac{1}{p^s}\right)^{-1}$$

$$\xi(s) = \frac{1}{2} s(s-1)\Gamma\left(\frac{s}{2}\right)\zeta(s) = \xi(1-s)$$

$$\zeta(s) = \chi(s)\,\zeta(1-s)$$

$$\chi(1-s) = 2(2\pi)^{-s}\,\Gamma(s)\cos\left(\frac{\pi s}{2}\right)$$

$$|\chi(1-s)| \sim \left(\frac{t}{2\pi}\right)^{\sigma - 1/2}$$

$$\xi(s) = e^{A+Bs}\prod_\rho \left(1 - \frac{s}{\rho}\right)^{s/\rho} \qquad \rho = \beta + i\gamma \qquad \zeta(\rho) = 0$$

RH: $\beta = 1/2$

RH $\Rightarrow$ Lindelöf : $\zeta(\frac{1}{2}+it) \ll_\varepsilon t^\varepsilon$

Moments :
$$\frac{1}{T}\int_0^T |\zeta(\tfrac{1}{2}+it)|^2\,dt \sim \log T \qquad (H+L)$$
$$\frac{1}{T}\int_0^T |\zeta(\tfrac{1}{2}+it)|^4\,dt \sim \frac{1}{2\pi^2}\log^4 T \qquad (\text{Ingham})$$

Mean Value (Montgomery - Vaughan)
$$\int_0^T \left|\sum_{n=1}^{N} a_n n^{it}\right|^2 dt = \big(T + O(N)\big)\sum_{n=1}^{N} |a_n|^2$$

If $a_n \in \ell^2$

$$\int_0^T \left| \sum_{n=1}^{\infty} a_n n^{it} \right|^2 dt = \sum_n \big(T + O(n)\big) |a_n|^2$$

e.g.

$$\int_0^T \left| \sum_{n=1}^{N} a_n n^{it} \right|^2 dt = \sum a_n \bar{a}_m \int_0^T \left(\frac{m}{n}\right)^{it} dt$$

$$= T \sum |a_n|^2 + \sum_{n \neq m} a_n \bar{a}_m \frac{\left(\frac{m}{n}\right)^{iT} - 1}{i \log \frac{m}{n}}$$

$|a_m \bar{a}_n| \leq |a_m|^2 + |a_n|^2$

$$\sum_m |a_m|^2 \sum_{n \neq m} \frac{1}{\left|\log \frac{n}{m}\right|}$$

split

$n = m + h$

$$\sum_{h < \frac{n}{2}} \frac{1}{\log\left(1 + \frac{h}{m}\right)} \sim \sum_{h < \frac{n}{2}} \frac{m}{n} \ll N \log N$$

<u>Approximate equations</u>

$$\zeta(s) = \underset{A}{\sum_{n \leq X} \frac{1}{n^s}} + \underset{B}{\frac{X^{1-s}}{s-1}} + \underset{C}{O(X^{-\sigma})}, \qquad \begin{matrix} \sigma > 0 \\ x \gg t \end{matrix}$$

$$\zeta(s) = \sum_{n \leq X} \frac{1}{n^s} + \chi(s) \sum_{n \leq y} \frac{1}{n^{1-s}} + O\left(X^{-\sigma} + t^{\frac{1}{2} - \sigma} y^{\sigma - 1}\right)$$

$$X y = \frac{t}{2\pi}$$

Approximate functional equation

$$\int_0^T |\zeta(\tfrac{1}{2}+it)|^2\,dt = \int_0^T |A+B+C|^2\,dt$$

$$= \int_0^T |A|^2\,dt + \int_0^T |B|^2\,dt + \cdots$$

$$\int_0^T A\bar{B}\,dt \leqslant \left(\int_0^T |A|^2\,dt\right)^{1/2}\left(\int_0^T |B|^2\,dt\right)^{1/2}$$

So, $$\int_0^T |A|^2\,dt = \int_0^T \Big|\sum_{n\le X}\frac{1}{n^{\frac12+it}}\Big|^2\,dt$$

$$\overset{MV}{=} \sum_{n\le X}\left(\frac{T+O(n)}{n}\right) = T\log X + O(X+T)$$

$$= T\log X + O(X) \quad \text{since } X \gg T$$

$$\int_0^T |B|^2\,dt = \int_0^T \frac{X}{|s-1|^2}\,dt \ll X$$

$$\int_0^T |C|^2\,dt \ll \frac{T}{X}$$

So $X = T \Rightarrow \displaystyle\int_0^T |\zeta(\tfrac12+it)|^2\,dt \sim T\log T.$

<u>Exercise</u>

$$\frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty} \zeta(s+z)\Gamma(z)X^z\,dz = \sum_{n=1}^{\infty}\frac{1}{n^s}\,\frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty}\Gamma(z)\left(\frac{X}{n}\right)^z dz$$

$\|$ Go left

$$= \sum_{n=1}^{\infty}\frac{e^{-n/X}}{n^s}$$

Use this to prove H+L formula

$$\Gamma(1-s)X^{1-s} + \zeta(s) + \frac{1}{2\pi i}\int^{-a+i\infty} \quad \|$$

s it

$\int_0^T |\zeta(1+it)|^2\,dt \ll 1$

$s = a+it$

$a = 1/2$

## Dirichlet L-functions

$\chi$ primitive character mod $q$

$$\sum_{\chi \bmod q}^{*} |L(\tfrac{1}{2},\chi)|^2 \sim \phi^*(q) \log q$$

primitive characters

$$\phi^*(q) = \#\text{ of primitive characters mod } q$$

$$= \sum_{d|q} \mu\left(\frac{q}{d}\right) \phi(d)$$

[ If $p$ is odd, $q = p^k$. How to make characters mod $p^k$?

Let $g$ be a primitive root mod $p^k$

$\{g, g^2, \dots, g^{\varphi(p^k)}\}$ = a set of reduced residues mod $p^k$

$$\xi = e^{\frac{2\pi i r}{\varphi(p^k)}} \quad \text{root of unity}$$

$$\chi_{r,g}(g^m) = \xi^m \qquad \text{Is completely multiplicative.}$$

Primitive if $p \nmid r$.

Orthogonality: If $\gcd(mn, q) = 1$,

$$\sum_{\chi \bmod q} \chi(m)\overline{\chi(n)} = \begin{cases} \phi(q) & \text{if } m \equiv n \bmod q \\ 0 & \text{otherwise} \end{cases}$$

Prim

$$\sum^{*} \chi(m)\chi(n) = \sum_{d|q} \mu\left(\frac{q}{d}\right) \phi(d). \qquad \text{(Homework)}$$

Polya - Vinogradov : $\chi$ primitive mod $q$

$$\Longrightarrow \left| \sum_{n \leq X} \chi(n) \right| \ll \sqrt{q} \log q$$

Large sieve inequality

$$\sum_{q \leq Q} \frac{q}{\varphi(q)} \sum_{\chi \bmod q}^{*} \left| \sum_{n=1}^{N} a_n \chi(n) \right|^2 \ll (N+Q^2) \sum_{n=1}^{N} |a_n|^2$$

Good bound if $N \ll Q^2$.

<u>Approximate functional identities</u>

$$\xi(s) = G(s)L(s) = \xi(1-s)$$

$$\frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} \xi(z+\tfrac{1}{2}) \frac{dz}{z} = \sum_n \frac{a_n}{\sqrt{n}} W(n)$$

where $W(n) = \frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} G(z+\frac{1}{2}) n^{-z} \frac{dz}{z}$

$\|$

Go left to $\xi(\frac{1}{2}) + \frac{1}{2\pi i} \int_{-2-i\infty}^{-2+i\infty} \xi(z+\tfrac{1}{2}) \frac{dz}{z}$

$$= \xi(\tfrac{1}{2}) - \frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} \xi(\tfrac{1}{2}-z) \frac{dz}{z}$$

$$= \xi(\tfrac{1}{2}) - \frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} \xi(\tfrac{1}{2}+z) \frac{dz}{z}$$

$$\xi(1/2) = 2\sum_{n=1}^{\infty} \frac{a_n W(n)}{\sqrt{n}}$$

So

$$L(1/2) = 2\sum_{n=1}^{\infty} \frac{a_n W(n) G(1/2)}{\sqrt{n}}$$

$$\xi(s,\chi) = \left(\frac{\pi}{q}\right)^{-s/2} \Gamma\left(\frac{s}{2}\right) L(s,\chi) = \varepsilon_\chi \xi(1-s,\bar{\chi}) \qquad \chi \text{ even}$$

$\uparrow$ Gauß sum

$$\xi(s,\chi)\xi(s,\bar{\chi}) = \frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty} \underbrace{\xi(z+\tfrac{1}{2},\chi)\xi(z+\tfrac{1}{2},\bar{\chi})}_{=:\Lambda(z+1/2)} \frac{dz}{z} \qquad (\Lambda(z) = \Lambda(1-.)$$

$$= \sum_{m,n}^{\infty} \frac{\chi(m)\overline{\chi(n)}}{\sqrt{mn}} W(mn)$$

$$|L(\tfrac{1}{2},\chi)|^2 = 2\sum_{m,n} \frac{\chi(m)\bar{\chi}(n)}{\sqrt{mn}} W\left(\frac{\pi mn}{q}\right) \qquad W(x) \approx 1 \text{ if } 0<x<1, \text{ small if } x>1$$

Average over $\chi$ mod $q$. $m=n$ $(mn,q)=1$

$$2\phi^*(q)\sum_m \frac{W\left(\frac{\pi m^2}{q}\right)}{m} \sim \phi^*(q)\log q$$

$m \neq n$

$$\sum_{d|q} \mu\left(\frac{q}{d}\right)\varphi(d) \sum_{m\equiv n\,(d)} \frac{W\left(\frac{\pi mn}{q}\right)}{\sqrt{mn}} \ll \frac{\sqrt{q}}{d}.$$