Catch-up #2:

General form of the conjectures for moments

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Conjectures

$$\frac{1}{T}\int_0^T |\zeta(\tfrac{1}{2}+it)|^{2k}\,dt \sim \frac{g_{k,U}\, a_{k,\zeta}}{k^2!} \log^{k^2} T$$

$$\frac{1}{\varphi^*(q)} \sum_{\chi(q)}^{*} |L(\tfrac{1}{2},\chi)|^{2k} \sim \frac{g_{k,U}\, a_{k,Dir}}{k^2!} \log^{k^2} q$$

$$\frac{1}{D^*} \sum_{d\le D}^{*} L(\tfrac{1}{2},\chi_d)^k \sim \frac{g_{k,Sp}\, a_{k,Jac}}{\frac{k(k+1)}{2}!} \log^{\frac{k(k+1)}{2}} D$$

$$\frac{1}{\dim(S_\ell(\Gamma_0(N)))} \sum_{f\in H_\ell(N)} L(\tfrac{1}{2},f)^k$$

$$\sim \begin{cases} \dfrac{g_{k,O}\, a_{k,Lev}}{\frac{k(k-1)}{2}!} \log^{\frac{k(k-1)}{2}} N \\[2ex] \dfrac{g_{k,O}\, a_{k,wt}}{\frac{k(k-1)}{2}!} \log^{\frac{k(k-1)}{2}} \ell \end{cases}$$

The a_k are explicit

The g_k from averaging over the matrix group

c-wc ① F_3(4)

Example:

$$\frac{1}{D^*} \sum_{d \le D}^{*} L(\tfrac{1}{2}, \chi_d)^k \sim \frac{g_{k,Sp}\, a_{k,Jac}}{\frac{k(k+1)}{2}!} \log^{\frac{k(k+1)}{2}} D$$

where

$$a_{k,Jac} = \prod_p \frac{(1-\frac{1}{p})^{k(k+1)/2}}{1+\frac{1}{p}} \left(\frac{(1-\frac{1}{\sqrt{p}})^{-k} + (1+\frac{1}{\sqrt{p}})^{-k}}{2} + \frac{1}{p} \right)$$

$$g_{k,Sp} = \left(\tfrac{1}{2} k(k+1)\right)! \prod_{j=1}^{k} \frac{j!}{(2j)!} .$$

Note (Keating-Snaith):

$$\left\langle \Lambda(1)^k \right\rangle_{USp(N)} \sim \frac{g_{k,Sp}}{(\frac{1}{2}k(k+1))!} N^{\frac{k(k+1)}{2}}$$

Problems:

1) Why just multiply things together?
2) Impossible to check numerically.

$$MV = C_0 \log^{\frac{k(k+1)}{2}} D + C_1 \log^{\frac{k(k+1)}{2}-1} D + \cdots$$

$$C_1 \approx 1000\, C_0$$

C-u② $F_2(5)$

More precise (but not always correct)
Conjecture

L-function mean value = Polynomial in the log Conductor + $O(\text{Conductor}^{-\frac{1}{2}+\varepsilon})$

- Not quite
- RMT can only predict the leading term.

$$\frac{1}{T}\int_0^T |\zeta(\tfrac{1}{2}+it)|^4 dt = P_4(\log T) + O(T^{-\frac{1}{8}+\varepsilon})$$

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(Theorem of Heath-Brown.
Conrey explicitly computed the polynomial)

Shifted Moments

$$\frac{1}{D^*}\sum_{d\le D}^{*} L(\tfrac{1}{2}+\alpha_1, \chi_d) \cdots L(\tfrac{1}{2}+\alpha_k, \chi_d)$$

Non-RMT conjecture

$$\langle \Lambda(e^{-\alpha_1}) \cdots \Lambda(e^{-\alpha_k}) \rangle_{USp(N)}$$

Theorem

The shifted moments have identical structure.

This gives a conjecture for the whole main term, which checks out numerically.