ROCHESTER SCHOOL, HOMEWORK 3: MATRICES

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In this homework set, the reference [C] refers to Conrey's article on pages 111–145 of the book *Recent Perspectives in Random Matrix Theory and Number Theory*, edited by Francesco Mezzadri and Nina Snaith.

The random matrices which are relevant to number theory are unitary matrices. We recall the definitions and basic properties.

If $X = (x_{j,k})$ then X^* or $X^\dagger$ is the matrix with entries $(\overline{x}_{k,j})$, where $\overline{z}$ is the complex conjugate of z. That is, X^* is the conjugate transpose of X. Physicists tend to write $X^\dagger$ instead of X^*. By definition, the $N \times N$ matrix X is *unitary* if $XX^* = I_N$, where I_N is the $N \times N$ identity matrix. That is, $X^{-1} = X^*$. The group of $N \times N$ unitary matrices is denoted $U(N)$.

The $N \times N$ matrix X is *orthogonal* if $XX^t = I_N$, where X^t is the transpose of X. That is, an orthogonal matrix is a real unitary matrix. The group of $N \times N$ orthogonal matrices is denoted $O(N)$, and the group of orthogonal matrices with determinant 1 is denoted $SO(N)$.

The $2N \times 2N$ unitary matrix X is *symplectic* if $XZ_NX^t = Z_N$, where

$$Z_N = \begin{pmatrix} 0 & I_N \\ -I_N & 0 \end{pmatrix}. \tag{1}$$

The group of $2N \times 2N$ unitary symplectic matrices is denoted $USp(2N)$, although sometimes people leave off the U and/or the 2.

Exercise 1. Show that the following are equivalent:

(1) $X \in U(N)$
(2) The columns of X are an orthonormal basis for $\mathbb{C}^N$

Thus, one way to make a random matrix in $U(N)$ is to start with any old random matrix and then apply Gram-Schmidt to the columns. Details during the school.

Exercise 2. Show that if $X \in U(N)$ then all eigenvalues of X have absolute value 1, and any finite collection of points on the unit circle arises as the set of eigenvalues of a unitary matrix.

Exercise 3. Formulate and prove the analogue of Exercise 1 for $SO(N)$. If N is odd, does there have to be an eigenvalue equal to $+1$?

Since $U(N)$ is a group, there exists a measure which is invariant under multiplication by elements of the group. That measure is called *Haar measure* and it is defined up to a multiplicative constant. Since $U(N)$ is compact, we usually normalize Haar measure so that the total volume of the group is 1. The orthogonal and symplectic groups each have their own Haar measure, but note that those measures do not come from restricting Haar measure on $U(N)$.

Since we usually are concerned with the eigenvalues, we usually express Haar measure in terms of the eigenvalues. See Section 3 of [C]. The main feature of Haar measure is *quadratic repulsion*: the eigenvalues do not like to be close to each other, and the measure vanishes quadratically as two eigenvalues become close to each other. Make sure that you can see this in the formula at the bottom of page 115.

Exercise 4. One way to understand the repulsion is to look at the unitary matrices which have a repeated eigenvalue. That (obviously?) is a set of measure zero in $U(N)$, but how small of a set is it? Naïvely, one might think that having a repeated eigenvalue is a codimension 1 condition, since it seems like you are just putting one constraint on the system. Show that having a repeated eigenvalue of a unitary matrix is actually a codimension > 1 condition.

Imagine a continuously varying 1-parameter family of unitary matrices. As the matrices change, the eigenvalues will move around on the unit circle. Unless you choose the matrices in a special way, the eigenvalues will not bump into each other, and you will never get a repeated eigenvalue. This phenomenon is known as "avoided crossing" and there is a picture of this on the cover of Peter Lax's Linear Algebra texbook.

Another matrix which features prominently is the *Vandermonde determinant*:

$$\Delta(z_1,\ldots,z_N)=\det(z_k^{j-1})_{jk}=\begin{vmatrix}1 & z_1 & z_1^2 & \cdots & z_1^N\\ 1 & z_2 & z_2^2 & \cdots & z_2^N\\ & \vdots & & \ddots & \vdots\\ 1 & z_N & z_N^2 & \cdots & z_N^N\end{vmatrix}. \tag{2}$$

This is not a random matrix, but it appears all over the place in random matrix theory (and elsewhere). It has a simple evaluation given in the next exercise.

Exercise 5. Show that $\Delta(z_1,\ldots,z_N)=\prod_{1\le j<k\le N}(z_k-z_j)$.

Suggestion: Argue that Δ is a polynomial in the z_j, and determine its degree. Explain why $\Delta=0$ when any two of the variables are equal. This will give you a whole bunch of factors of the polynomial. If you have all the factors, then your polynomial only needs an overall scale factor. You can determine that by looking at a limiting case or some particular coefficient.

Note that Haar measure on $U(N)$ can be written in terms of a Vandermonde in the eigenvalues.

Since elementary row/column operations change the determinant of a matrix in a predictable way, the Vandermonde can arise in many disguised forms.

Exercise 6. First prove that if k is a positive integer then

$$\cos(kx)=2^{k-1}\cos^k(x)+\textit{lower degree terms in } \cos(x). \tag{3}$$

Then using only that fact and basic properties of determinants show that

$$\begin{vmatrix}1 & \cos(z_1) & \cos(2z_1) & \cdots & \cos(Nz_1)\\ 1 & \cos(z_2) & \cos(2z_2) & \cdots & \cos(Nz_2)\\ & \vdots & & \ddots & \vdots\\ 1 & \cos(z_N) & \cos(2z_N) & \cdots & \cos(Nz_N)\end{vmatrix}=2^{N(N-1)/2}\Delta(\cos(z_1),\ldots,\cos(z_N)). \tag{4}$$

Then do the analogous determinant where all the cosines are replaced by sines.