ROCHESTER SCHOOL, HOMEWORK 2: THE EULER PRODUCT

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These exercises are meant to be useful to people who are unfamiliar with this material, not a burden to those who have seen it before. So please skip anything that strikes you as easy and, conversely, work carefully through anything that looks new to you.

The point of these exercises is to prove the Euler product formula, which says if $\mathfrak{Re}(s) > 1$, then

$$\sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p \text{ prime}} \left(1 - p^{-s}\right)^{-1}. \tag{1}$$

Exercise 1 (Messy, but trivial). Show that

$$\left(1 - \frac{1}{2^s}\right)^{-1} \left(1 - \frac{1}{3^s}\right)^{-1} = 1 + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \frac{1}{6^s} + \frac{1}{8^s} + \frac{1}{9^s} + \frac{1}{12^s} + \frac{1}{16^s} + \frac{1}{18^s} + \dots.$$

Exercise 2 (The first key step). Let

$$\mathcal{S}(X) = \{n \ : \ \text{ if a prime } p \text{ divides } n\text{, then } p \le X\}.$$

The set $\mathcal{S}(X)$ is called the set of X-smooth numbers. Show that

$$\prod_{\substack{p \text{ prime} \\ p \le X}} \left(1 - p^{-s}\right)^{-1} = \sum_{n \in \mathcal{S}(X)} \frac{1}{n^s}.$$

Note that putting $X = 3$ yields the result of exercise 1 above.

Exercise 3 (The second key step). Show that if $s = \sigma + it$ with $t \in \mathbb{R}$ and $\sigma > 1$, then

$$\left| \sum_{n=1}^{\infty} \frac{1}{n^s} - \sum_{n \in \mathcal{S}(X)} \frac{1}{n^s} \right| \le \sum_{n > X} \frac{1}{n^\sigma},$$

hence the difference tends to zero as $X \to \infty$. This proves that equation (1) holds true for $\mathfrak{Re}(s) = \sigma > 1$.

Exercise 4. It can be said that the "1" in $\frac{1}{n^s}$ in equation (1) is the fundamental theorem of arithmetic (that every positive integer can be decomposed *uniquely* into prime powers). Why is this?

The Euler product generalizes to other situations. An arithmetic function is a function whose domain is the positive integers. An arithmetic function $a(n)$ is said to be *completely multiplicative* or *totally multiplicative* if $a(mn) = a(m)a(n)$ for any positive integers m and n. An example of a totally multiplicative function would be a Dirichlet character.

Exercise 5. Repeat the above exercises to show that if $a(n)$ is a completely multiplicative arithmetic function, and $|a(n)| = O(1)$ then if $\mathfrak{Re}(s) > 1$,

$$\sum_{n=1}^{\infty} \frac{a(n)}{n^s} = \prod_{p \text{ prime}} \left(1 - \frac{a(p)}{p^s}\right)^{-1} . \tag{2}$$

Exercise 6. If $a(n)$ got large for large n then the sum or product in equation (2) might not converge for $\mathfrak{Re}(s)$ near 1. For example if $a(n) = n$, then $\mathfrak{Re}(s) > 2$ is required. (Why?) Show that if $\nu \geq 0$ and $|a(n)| = O\left(n^{\nu+\epsilon}\right)$ for any $\epsilon > 0$ then $\mathfrak{Re}(s) > \nu + 1$ is needed to ensure convergence of both the sum and product in equation (2).

An arithmetic function $b(n)$ is said to be *multiplicative* if $b(mn) = b(m)b(n)$ for any coprime positive integers m and n (that is, if the only positive integer which divides both m and n is the number 1).

Exercise 7. Show that if $b(n)$ is a multiplicative function with $b(n) = O\left(n^{\epsilon}\right)$ for any $\epsilon > 0$, and if $\mathfrak{Re}(s) > 1$, then

$$\sum_{n=1}^{\infty} \frac{b(n)}{n^s} = \prod_{p \text{ prime}} \left(\sum_{j=0}^{\infty} \frac{b(p^j)}{p^{js}}\right) . \tag{3}$$

Exercise 8. If $b(n)$ is not just multiplicative, but completely multiplicative, why is the right hand side of equation (3) the same as the right hand side of equation (2)?

Exercise 9. An important class of functions in number theory are *cusp forms.* The simplest example is

$$\begin{aligned} \Delta(z) &= q \prod_{n=1}^{\infty} (1 - q^n)^{24} \\ &= \sum_{n=1}^{\infty} \tau(n) q^n , \end{aligned}$$

where $q = e^{2\pi i z}$. The arithmetic function $\tau(n)$ defined above satisfies some surprising properties:

- $\tau(n)$ is multiplicative and satisfies $\tau(p^{j+2}) = \tau(p)\tau(p^{j+1}) - p^{11}\tau(p^j)$ for $j \geq 0$.
- $|\tau(p)| \leq 2p^{11/2}$.

(Both these results were conjectured by Ramanujan. The first was proved by Mordell in 1917, the second by Deligne in 1974). Assuming these facts, show that if $\mathfrak{Re}(s) > 13/2$, then the L-function associated to Δ satisfies

$$\begin{aligned} L_{\Delta}(s) &= \sum_{n=1}^{\infty} \frac{\tau(n)}{n^s} \\ &= \prod_{p} \left(1 - \frac{\tau(p)}{p^s} + \frac{p^{11}}{p^{2s}}\right)^{-1} . \end{aligned}$$