ROCHESTER SCHOOL, HOMEWORK 1: POISSON SUMMATION

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In this homework set, the reference [H-B] refers to Roger Heath-Brown's lectures on pages 1–30 of the book *Recent Perspectives in Random Matrix Theory and Number Theory*, edited by Francesco Mezzadri and Nina Snaith. By now you should have bought this book, and don't forget to bring your receipts to the school so we can reimburse you (up to $65).

These exercises are meant to be useful to people who are unfamiliar with this material, not a burden to those who have seen it before. So please skip anything that strikes you as easy, and conversely, work carefully through anything that looks new to you.

Recall the Poisson summation formula: if

$$\hat{f}(t) := \int_{-\infty}^{\infty} f(x) e^{-2\pi i x t} dx \tag{1}$$

then

$$\sum_n \hat{f}(n) = \sum_n f(n). \tag{2}$$

See Theorem 7 of [H-B] for sufficient conditions on f.

Be warned that there are several different normalizations of the Fourier transform, and you almost always have to explicitly write out what you mean each time you use it. Obviously the normalization affects the inverse transform. For this particular one, the inverse transform is

$$f(x) = \int_{-\infty}^{\infty} \hat{f}(t) e^{2\pi i x t} dt. \tag{3}$$

If you like to use Mathematica, then to use the above definition you have to override the default for `FourierTransform` by giving the setting `FourierParameters -> {0, -2 Pi}`. (And Mathematica says that this is the normalization used in "signal processing"!) Maple's default Fourier transform, `inttrans[fourier]`, is defined as

$$F(t) = \int_{-\infty}^{\infty} f(x) e^{-ixt} dx, \qquad , \qquad f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(t) e^{ixt} dt \tag{4}$$

and there is no way to override the default.

Exercise 1. With the Fourier transform defined as in (1), show that $e^{-\pi x^2}$ is its own Fourier transform.

Exercise 2. With what normalization is the Gaussian density function, $\frac{1}{\sqrt{2\pi}} e^{-x^2/2}$, its own Fourier transform?

Exercise 3. Show that if $g(x) = f(kx)$ then $\hat{g}(t) = \frac{1}{k} \hat{f}(t/k)$.

Exercise 4. Combine Exercises 1 and 3 to derive the transformation law for the θ-function given at the bottom of page 9 of [H-B].

Exercise 5. Work through the analytic continuation and functional equation of the Riemann ζ-function on pages 9 and 10 of [H-B].

Note that the ψ-function on page 10 has no relation to the prime counting function usually denoted $\psi(x)$. Heath-Brown is just following Riemann's original notation. Titchmarsh, in *The Theory of the Riemann Zeta-function* uses the same notation in the "third method" for proving the functional equation (page 21).

If you don't have a copy of the Newton Institute proceedings, you should be able to do the first three problems just using the definition of Fourier transform. For the last two problems, you can instead refer to section (2.6) of Titchmarsh's *The theory of the Riemann zeta-function.* The transformation law for the θ-function is given in formula (2.6.3), and the functional equation for the zeta-function is derived in a similar way to Heath-Brown. (The derivation in Titchmarsh is mathematically equivalent but not as cleanly presented as in Heath-Brown).