FUNCTIONAL DIFFERENTIAL EQUATIONS WITH STATE-DEPENDENT DELAYS: THEORY AND APPLICATIONS

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1. Introduction

Studies of single differential equations with state-dependent delayed or advanced arguments go back at least to Poisson [187], but as an object of a broader mathematical activity the area is rather young.

Most work during the past 50 years is devoted to equations with state-dependent delays, which arise as models in applications. A prominent example is the two-body problem of electrodynamics, which remained a mathematical terra incognita until R. D. Driver's work began to appear in the sixties of the last century.

The present survey reports about the more recent work on equations with state-dependent delays, with emphasis on particular models and on the emerging theory from the dynamical systems point of view. Several new results are presented. It will also become obvious that challenging problems remain to be solved.

State-dependent delays were addressed earlier in survey papers on the larger area of functional differential equations, notably by Halanay and Yorke [96] and Myshkis [169]. It is tempting to borrow as a motto from Halanay and Yorke [96] their nice statement

> *This ... proved once more, if necessary, that the delay existing at the present time between the moment results are obtained and the moment of publication (as well as the great number of publications which are very difficult to follow) makes it necessary to present from time to time such reports on yet unpublished results and unsolved problems.*

The simplest example of a differential equation with constant delay is the linear equation

$$y'(t) = a\,y(t-h)$$

with the fixed delay $h > 0$ and a parameter $a \in \mathbb{R}$. Analogues with state-dependent delay like

$$x'(t) = a\,x(t - r(x(t))),$$

with a bounded delay map $r : \mathbb{R} \to [0, h]$, are already nonlinear in general. Both equations can be written in the same general form

$$x'(t) = f(x_t) \tag{1.0.1}$$

of a delay differential equation. Here $f : U \to \mathbb{R}^n$ is defined on a subset U of the set $(\mathbb{R}^n)^{[-h,0]}$ of all functions $\phi : [-h, 0] \to \mathbb{R}^n$. The *solution segment* $x_t : [-h, 0] \to \mathbb{R}^n$ is given by

$$x_t(s) = x(t + s), \ -h \le s \le 0.$$

In case of the examples above, we have

$$n = 1,\ U = \mathbb{R}^{[-h,0]},\ f(\phi) = a\,\phi(-h) \text{ and } f(\phi) = a\,\phi(-r(\phi(0))),\ \text{respectively.}$$

The maps $f : U \to \mathbb{R}^n$ describing equations with state-dependent delay have in general less smoothness properties than those representing equations with constant delay, and the theory of Retarded Functional Differential Equations (RFDEs) which has been developed since the fifties of the last century (see, e. g., [100, 55]) is not applicable to equations with state-dependent delay. This concerns already basic questions of existence, uniqueness and smooth dependence on initial data for the initial value problem (IVP)

$$x'(t) = f(x_t),\ x_{t_0} = \phi \tag{1.0.2}$$

which for data $\phi \in U$ and for $t_0 \in \mathbb{R}$ is associated with Equation (1.0.1). When in the sequel we speak of a solution $x : [t_0 - h, T) \to \mathbb{R}^n$, $t_0 < T \le \infty$, to the IVP (1.0.2) it is always understood that x at least satisfies

$$x_t \in U \text{ for all } t \in [0, T), \; x_{t_0} = \phi,$$

and that x is differentiable on (t_0, T) with

$$x'(t) = f(x_t) \text{ for all } t \in (0, T).$$

In Section 3 solutions of well-posed IVPs will even be continuously differentiable.

The following Section 2 describes examples of differential equations with state-dependent delays which arise in physics, automatic control, neural networks, infectious diseases, population growth, and cell production. Some of these models differ considerably from others, and most of them do not look simple. Typically the delay is not given explicitly as a function of what seems to be the natural state variable; the delay may be defined implicitly by a functional, integral or differential equation and should often be considered as part of the state variables.

Modelling systems with state-dependent delays seems to require extra care, perhaps because there is not much experience with this phenomenon. We tried to avoid models for which as yet no consistent motivation can be given.

The models in Section 2 indicate the types of equations to which the subsequent sections are confined. Not covered are, for example, nonautonomous systems, delays of the form $r = r(t) = a\,t + b\,x(t)$ or $r(t) = a\,t + b\,x''(t)$ like in [193, 194, 195], and constructions of explicit solutions as in [106]. Also we do not say much about state-dependent delays in control theory. Early work in this area is found in [81, 82, 83, 163].

Section 3 presents a framework for the study of the IVP (1.0.2). We analyze how state-dependent delays prevent the IVP (1.0.2) from being well-posed on open subsets U of familiar Banach spaces, and see that under mild smoothness hypotheses the IVP (1.0.2) *is* well-posed for data only in a submanifold of finite codimension. This *solution manifold* is given by the equation considered and generalizes the familiar domain of the generator of the semigroup given by a linear autonomous RFDE (as in [100, 55]). On the solution manifold the IVP (1.0.2) with $t_0 = 0$ defines a semiflow of continuously differentiable solution operators. This resolves the problem of linearization for equations with state-dependent delay, which had been pointed out earlier by Cooke and Huang [49]. The widely known heuristic technique of *freezing the delay at equilibrium and then linearizing the resulting RFDE*, often skilfully applied, can now be understood in terms of the true linearization. It is clarified why familiar characteristic equations for linear autonomous RFDEs can be used to analyze local dynamics generated by differential equations with state-dependent delay.

At stationary points the continuously differentiable solution operators have local center, stable, and unstable manifolds. It is shown that these stable and unstable manifolds of maps yield local stable and unstable manifolds also for the semiflow. In particular there is a convenient Principle of Linearized Stability. Center manifolds for the semiflow can not be immediately obtained as just described; they are constructed in Section 4.

Examples to which the results of Section 3 apply are given in [213, 215]; for the proof in [215] that hyperbolic stable periodic orbits exist the smoothness results of

Section 3 are indispensable. Most other work about which we report was accomplished before the basic theory of Section 3 was developed; some further work was done in parallel. We do not attempt to present these other results in the framework of Section 3. In fact, for many models it remains to be studied whether they fit into this framework or not.

In Section 4 a new result is proved, namely existence of Lipschitz continuous local center manifolds for the semiflow found in Section 3, at stationary points. The more technical proof that these center manifolds actually are continuously differentiable will appear elsewhere. An important open problem is to obtain more smoothness, as it was established for local unstable manifolds [131].

Section 5 is about local Hopf bifurcation, i.e., about the appearance of small periodic orbits close to a stationary point when a parameter in the underlying differential equation is varied and passes a critical value. We state a Hopf bifurcation theorem recently obtained by M. Eichmann [66], which seems to be the first such result for differential equations with state-dependent delays.

Section 6 presents results about differentiability of solutions with respect to parameters and initial data, for a certain class of nonautonomous differential equations with state-dependent delay. The framework is different from the one developed in Section 3. The IVP is considered for Lipschitz continuous initial data, and a quasi-normed space derived from Sobolev spaces turns out to be useful for studying differentiability under relaxed smoothness assumptions, which may be convenient for applications.

Section 7 deals with periodic orbits. The search for periodic solutions has been an important topic in the study of nonlinear autonomous delay differential equations since the sixties of the last century. By now, several methods have been developed in this area. The most general results on existence and global bifurcation employ fixed point theorems and the fixed point index. Others are based on the study of 2-dimensional invariant sets, or on Poincaré-Bendixson type analysis of plane curves which are obtained from evaluations like $x_t \mapsto (x(t), x(t-1))$ along certain solutions. There are local and global Hopf bifurcation theorems for RFDEs, the Fuller index counting periodic orbits is used, and certain symmetric periodic solutions can be obtained from associated ordinary differential systems. Not all approaches mentioned here have been tried for equations with state-dependent delays, which cause complications. We describe results which use the topological concept of ejectivity, and an approach which yields stable periodic orbits. Let us add here that topological tools (fixed point theorems, degree, coincidence degree) have also been employed to prove existence of periodic solutions to nonautonomous, *periodic* differential equations with state-dependent delay [44, 140, 141, 228, 232]; related work on nonautonomous equations is found in [51, 56].

The topic of Section 8 is limiting behaviour with respect to the independent variable and with respect to parameters. Subsection 8.1 presents a study of a two-dimensional attractor with periodic orbits, which extends a part of a result for equations with constant delay. Subsection 8.2 reports about results of Mallet-Paret and Nussbaum on the precise asymptotic shape of periodic solutions in a singular perturbation problem. These results are genuine for equations with state-dependent delay, and involve work on unusual eigenvalue problems for so-called *max-plus operators*. Subsection 8.3 describes an approach to periodic solutions in case of small delay. Subsection 8.4 contains an application of the monotone dynamical systems theory which yields a generic convergence result, and Subsection 8.5 comments on further results about stability and oscillation properties.

Section 9 deals with numerical methods. The study of numerical approximation of solutions to differential equations with state-dependent delay goes back at least to the mid-sixties of the last century, and since then it has been an intensively investigated area. The section begins with a brief summary of continuous Runge-Kutta methods for ordinary differential equations and reports about modifications and extensions which are necessary in case of state-dependent delays.

Let us mention here a few out of many open questions, in addition to those addressed in the subsequent sections. Do equations with state-dependent delay generate semiflows with better smoothness properties than obtained in Section 3, on suitable invariant sets ? The results from [131] on unstable manifolds point in this direction. A suspicion is that periodic solutions of equations with state-dependent delay may have stronger stability properties than their counterparts in related equations with constant delay. Can this be made precise and established, for suitable classes of equations ? Complicated motion, like chaos, has not yet been rigorously shown to exist for equations with state-dependent delay. Very little is known about the general two-body problem of electrodynamics with two charged particles in a 3-dimenson configuration space. Vanishing state-dependent delays, like in a collision in the two-body problem, are also limiting cases of advanced arguments; this indicates that a better understanding of more general differential equations with both delayed and advanced state-dependent arguments may be needed.

It is convenient to end this introduction with notation, for function spaces which occur frequently in the sequel. The Banach spaces of continuous, Lipschitz continuous, and continuously differentiable maps $\phi : [-h, 0] \to \mathbb{R}^n$ are denoted by

$$C = C([-h,0];\mathbb{R}^n),\ C^{0,1} = C^{0,1}([-h,0];\mathbb{R}^n),\ \text{and}\ C^1 = C^1([-h,0];\mathbb{R}^n),$$

respectively. The norms on these spaces are given by

$$\|\phi\|_C = \max_{-h\le t\le 0}|\phi(t)|, \|\phi\|_{C^{0,1}} = \|\phi\|_C + \sup_{t\ne s}\frac{|\phi(t)-\phi(s)|}{|t-s|}, \|\phi\|_{C^1} = \|\phi\|_C + \|\phi'\|_C,$$

respectively.

If X and Y are real or complex Banach spaces then $L(X,Y)$ denotes the Banach space of continuous linear mappings $T : X \to Y$, with the norm given by $\|T\| = \sup_{\|x\|\le 1}\|Tx\|$.

2. Models and applications

A remark in [213] says that *state-dependent delays arise in various circumstances, but it seems not obvious how to single out a tractable class of equations which contains a large set of examples which are well motivated.* The difficulty of singling out a tractable class of equations to include many interesting models may prove to be an extremely valuable source to stimulate new mathematical techniques and theories. In this section we describe differential equations with state-dependent delay that arise from electrodynamics, automatic and remote control, machine cutting, neural networks, population biology, mathematical epidemiology and economics.

2.1. **A two-body problem of classical electrodynamics.** In Driver [59] (see also [58, 60]), a mathematical model for a two-body problem of classical electrodynamics incorporating retarded interaction is proposed and analyzed. He considers the motion for two charged particles moving along the x-axis and substituted the expressions for the field of a moving charge, calculated from the Liénard–Wiechert potential, into the Lorentz-Abraham force law. Radiation reaction is omitted, but time delays are incorporated due to the finite speed of propagation, c, of electrical effects. As a result, the model is a system of delay differential equations involving time delays, which depend on the unknown trajectories. From this model and after some analysis, he obtains a system of six delay-differential equations for the evolution of the states, the velocities and the time delays.

To describe his model, we denote by $x_i(t)\,(i = 1, 2)$ the positions of the two point charges on the axis in a given inertial system at time t, the time of an observer in that system. Let $v_i(t) = x_i'(t)\,(i = 1, 2)$ be the velocities of the charges. As mentioned above, we omit radiation reaction but allow an external electric field, $E_{ext}(t, x)$, in the x-direction, that is assumed to be continuous over some open set D in the (t, x)-plane. Then the equation of motion of charge i is

$$\frac{m_i v_i'(t)}{[1 - v_i^2(t)/c^2]^{3/2}} = q_i E_j(t, x_i(t)) + q_i E_{ext}(t, x_i(t)),\ \ i, j \in \{1, 2\}, j \neq i, \tag{2.1.1}$$

where m_i is the rest mass and q_i is the magnitude of charge i, c is the speed of light, and $E_j(t, x)$ is the electric field at (t, x) due to other charge $j \neq i$. The magnetic field of charge j is not involved in this one-dimensional case.

The field at time t and at the point $x_i(t)$ produced by charge j is assumed to be that computed from the Liénard-Wiechert potentials. The expression for this field involves a time lag, $t - \tau_{ji}$, representing the instant at which a light signal would have to leave charge j in order to arrive at $x_i(t)$ at the instant t. Therefore, the delay $\tau_{ji}(t)$ must be a solution of the functional equation

$$\tau_{ji}(t) = |x_i(t) - x_j(t - \tau_{ji}(t))|/c. \tag{2.1.2}$$

Clearly, $\tau_{ji}(t)$ cannot be written explicitly.

Because of the occurrence of time delays in the model equation (2.1.1), one needs to specify initial trajectories of the two charges over some appropriate interval $[\alpha, t_0]$. Consider now those initial trajectories and their extensions $(x_1(t), x_2(t))$ defined on some interval $[\alpha, \beta)$, where $\beta > t_0$, such that

(a) each $x_i'(t)$ is continuous and $|x_i'(t)| < c$ for all $t \in [\alpha, \beta)$;
(b) $x_2(t) > x_1(t)$ and $(t, x_i(t)) \in D$ for all $t \in [t_0, \beta)$;
(c) the two functional equations $\tau_{ji}^0 = |x_i(t_0) - x_j(t_0 - \tau_{ji}^0)|/c$ have solutions $\tau_{ji}^0, i \neq j, i, j \in \{1, 2\}$.

Then Driver proves that $(x_1(t), x_2(t))$ is a solution of (2.1.1)-(2.1.2) if and only if it satisfies the following system of six delay differential equations for $t \in (t_0, \beta)$:

$$\begin{cases} x_i'(t) = v_i(t), \\ \tau_{ji}'(t) = \frac{(-1)^i v_i(t) - (-1)^i v_j(t-\tau_{ji}(t))}{c-(-1)^i v_j(t-\tau_{ji}(t))}, \\ \frac{v_i'(t)}{[1-v_i^2(t)/c^2]^{3/2}} = \frac{(-1)^i a_i c}{\tau_{ji}^2(t)} \cdot \frac{c+(-1)^i v_j(t-\tau_{ji}(t))}{c-(-1)^i v_j(t-\tau_{ji}(t))} + q_i E_{ext}(t, x_i(t))/m_i, \end{cases} \tag{2.1.3}$$

where $\tau_{ji}(t_0) = \tau_{ji}^0, a_i = q_1q_2/(4\pi\epsilon_0 m_i c^3)$ (a constant, and in particular, ϵ_0 is the dielectric constant of free space), and $(i, j) = (1, 2)$ or $(2, 1)$.

It is shown in Driver [59] that if given initial trajectories satisfy condition (a) for $\alpha \le t \le t_0$, condition (b) at t_0, and condition (c), and if $E_{ext}(t, x)$ is Lipschitz continuous with respect to x in each compact subset of D and if the initial velocity of each particle is Lipschitz continuous, then a unique solution does exist. This solution can be continued as long as the charges do not collide ($\lim x_1(t) = \lim x_2(t)$ as t approaches the right endpoint of the maximal interval for existence) and neither $(t, x_1(t))$ nor $(t, x_2(t))$ approaches the boundary D.

We remark here that in Driver and Norris [65], the above Lipschitz continuity for the initial velocities is relaxed to the integrability of the initial velocity on $[\alpha, t_0]$. In Driver [62], one special case was given where the positions and velocities of the particles *at some instant* will determine the state of the system. More precisely, in this example of electrodynamic equations of motion, instantaneous values of positions and velocities of the particles will determine their trajectories, if the solutions are defined for all future time. This property was frequently conjectured, asserted, or implicitly assumed, as in Newtonian mechanics and as indicated by the long list of related references in Driver [62], but this property should not be expected for general electrodynamic equations.

In the case where $E_{ext}(t, x) = 0$ for all $(t, x) \in \mathbb{R}^2$ and if $q_1q_2 > 0$ (two point charges of like sign), then $\lim_{t\to\infty}[x_2(t) - x_1(t)] = \infty$ and $|v_i(t)| \le \bar{c} < c$ for all $t \ge \alpha$. This is a quite interesting result as it indicates that the delay $\tau_{ji}(t)$ may become unbounded, as such, one obtains a system of functional differential equation with unbounded state-dependent delays.

It is noted that if three-dimensional motions are considered, then one obtains a functional differential system of neutral type where the delays are dependent on the states, and the change rate of v_i at the current time also depends on its historical value $v_j'(t - \tau_{ji})$. More precisely, if we introduce a unit vector

$$u_i = \frac{x_i - x_j(t - \tau_{ji})}{c\tau_{ji}}$$

and a scalar quantity

$$\gamma_{ij} = 1 - \frac{1}{c} v_j(t - \tau_{ji}) \cdot u_i$$

as Driver [63] does, where $\cdot$ indicates the dot or scalar product in $\mathbb{R}^3$ (note, of course, x_1, x_2 are now vectors in $\mathbb{R}^3$), then the Lorentz force law yields

$$v_i'(t) = \frac{q_i(1 - |v_i|^2/c^2)^{1/2}}{m_i}[E_j + (v_i/c \cdot E_j)(u_i - v_i/c) - (v_i/c \cdot u_i)E_j], \tag{2.1.4}$$

where E_j is the retarded (vector-valued) electric field arriving at x_i at the instant t from particle j. This field, in $\mathbb{R}^3$, can be found from the Liénard-Weichert potentials as

$$E_j = \frac{kcq_j}{\tau_{ji}^2\gamma_{ij}^3}[u_i - v_i(t-\tau_{ji})/c][1 - |v_j|^2(t-\tau_{ji})] \tag{2.1.5}$$
$$+\frac{kq_j}{\tau_{ji}\gamma_{ij}^3}u_i \times ([u_i - v_j(t-\tau_{ji})/c] \times v_j'(t-\tau_{ji})),$$

where $k > 0$ is a constant depending on the units, and $\times$ indicates the vector cross product in $\mathbb{R}^3$. The dynamical adaptation for τ_{ji} is given by

$$\tau_{ji}'(t) = \frac{u_i \cdot [v_i - v_j(t-\tau_{ji})]}{c\gamma_{ij}}. \tag{2.1.6}$$

In the above discussions, the motion of each particle is influenced by the electromagnetic fields of the others, and due to the finite speed of the propagation of these fields, the model equations describing the motion of charged particles via action at a distance will involve time delays which depends on the state of the whole system. In Driver [64] and in Hoag and Driver [113], it is noted that if one considers that the basic laws of physics are symmetric with respect to time reversal, then the existence of these delays implies that there should also be advanced terms in the equations, and thus one is led to a system of functional differential equations with mixed arguments (Hoag and Driver [113]), and of neutral type (Driver [64]).

In summary and in conclusion, despite the fact that much of the work by Driver and his collaborators on electrodynamics was published nearly 40 years ago, many interesting questions related to the fundamental issues of electrodynamics remain unsolved mathematically and Driver's models remain as a source of inspiration for the theoretical development and a testing tool for new results.

2.2. **Position control.** State-dependent delays arise naturally in automatic position control if in the feedback loop running times of signals between the object of study and a reference point are taken into account.

In [212], the following simple and idealized situation is considered: An object moves along a line and regulates its position relative to an obstacle by means of signals which are reflected by the obstacle. Let $x(t)$ denote the position of the object at time $t \in \mathbb{R}$. The aim of control is that the object should not collide with the obstacle located at $-w < 0$, and that the object should be close to a preferred position at distance w from the obstacle, i.e., at $0 \in \mathbb{R}$. The difficulty for the control is that measurement of the position via signal running times takes time during which the object is moving. More precisely, assume that signals travel from the object to the obstacle at a speed $c > 0$ and are reflected. The object senses the reflected signals and measures the signal running time $s(t)$ between the emission and detection at time t:

$$c\,s = |x(t - s(t)) + w| + |x(t) + w|.$$

Then it uses s to compute a distance d from the obstacle according to

$$d = \frac{c}{2}s.$$

This seems to be reasonable since it gives the true distance at time t at least if at times $t - s(t)$ and t the object is in the same position. We must however emphasize that in general d is only a *computed* length and not the true distance, and that we consider a situation where there is no direct, immediate access to the true position.

Depending on the computed distance $d-w$ from the preferred the object adjusts its speed in size and direction, with a reaction time lag $r>0$ which is assumed to be constant. This negative feedback mechanism is then described by the differential equation

$$x'(t) = v(d(t-r) - w)$$

where $v:\mathbb{R}\to\mathbb{R}$ represents negative feedback with respect to the preferred position $0\in\mathbb{R}$ in the sense that

$$\delta\, v(\delta) < 0 \text{ for all } \delta \neq 0$$

holds. Therefore, we are led to the system

$$x'(t) = v\left(\frac{c}{2}s(t-r) - w\right) \tag{2.2.1}$$

$$c\,s(t) = |x(t-s(t)) + w| + |x(t) + w| \tag{2.2.2}$$

for positive parameters c, w, r and a negative feedback nonlinearity v. Notice that for motion $x:\mathbb{R}\to\mathbb{R}$ with speed $|x'|$ bounded by a constant $b<c$, that is, slower than the signal speed, the equation

$$s = \frac{1}{c}(|x(t-s) + w| + |x(t) + w|)$$

equivalent to (2.2.2) has a unique solution $s=\sigma(x|_{(-\infty,t]})$ because for given x and t the right hand side of the equation defines a contraction $[0,\infty)\to[0,\infty)$. Then one can take the right hand side of equation (2.2.2) with $t-r-\sigma(x|_{(-\infty,t-r]})$ and $t-r$ instead of $t-s$ and t, respectively, and replace $c\,s(t-r)$ in Equation (2.2.1), which yields a single delay differential equation with state-dependent delay. Essentially the same reasoning shows that for Lipschitz continuous solutions $x:[-h,t_e)\to\mathbb{R}$, $h>0$ and $t_e>0$, which have Lipschitz constants strictly less than c and satisfy suitable boundedness conditions, the system (2.2.1-2) can be rewritten as a single delay differential equation of the form (1.0.1).

A closely related model, with an explicit expression for a fraction of the signal running time instead of equation (2.2.2) for the total signal running time, was mentioned earlier by Nussbaum [181]. A similar model was also proposed by Messer [164].

It is perhaps more realistic to replace the first order differential equation (2.2.1) with the constant reaction lag $r>0$ by Newton's law

$$x'' = A$$

with an instantaneous restoring force A which depends on the computed distance d. Also friction might be taken into account. Such a model was studied in [215], by means of the fundamental theory presented in the next section.

We describe the main results from [212, 215], namely existence of stable periodic orbits, in subsection 7.3.

In [32, 33] Büger and Martin study a case of velocity control which involves signal running times. They consider an object travelling along a line which, ideally, should have a certain prescribed constant velocity v_0 throughout its whole journey. The object regulates its velocity $v=x'$ by adjusting its acceleration $a=v'$ according to a negative feedback relation

$$a\cdot(v-v_0) < 0 \text{ for all } v\in\mathbb{R}\setminus\{v_0\},$$

that is, the object speeds up if $v<v_0$ and slows down when $v>v_0$. This strategy is remotely controlled by a base located at $x=0$ to which the object transfers its current velocity v with a certain transmission velocity $c_1 > 0$, and the negative feedback information is then transmitted back from the base to the object with a transmission velocity $c > 0$. The total signal running time of the signals emitted from the object in the past and transmitted back from the base to the object as to arrive at time t is given by

$$s = s_1 + s_2$$

where

$$s_1 = \frac{1}{c_1}|x(t-s)| \text{ and } s_2 = \frac{1}{c}|x(t)|.$$

If an additional constant reaction time $r > 0$ of the base is taken into account then the total transmission delay is

$$r + s(t)$$

where now

$$s(t) = \frac{1}{c_1}|x(t-r-s(t))| + \frac{1}{c}|x(t)|.$$

The preceding equation and the second order differential equation

$$x''(t) = A(x'(t-r-s(t)) - v_0)$$

with a negative feedback nonlinearity $A : \mathbb{R} \to \mathbb{R}$ constitute the model. Büger and Martin investigated a simplification which neglects the running time of the signal from the object to the base, in which case the model is reduced to the single equation

$$x''(t) = A(x'(t-r-\frac{1}{c}|x(t)|) - v_0) \tag{2.2.3}$$

with an explicitly given state-dependent delay. It is shown in [32, 33] that close to segments of constant velocity solutions $t \mapsto v_0 t + c$ there exist segments of solutions of Equation (2.2.3) for which $t - r - \frac{1}{c}|x(t)|$ remains bounded from above by some T. In this case the object reacts only to velocities achieved before $t = T$, which may not be adequate for larger velocities reached later. The phenomenon occurs for solutions whose speed $|v| = |x'|$ approaches the signal speed c or grows beyond. Büger and Martin call it the *escaping disaster*. In [33] they design another control mechanism which overcomes this kind of instability of the constant velocity solutions.

2.3. **Mechanical models.** State-dependent models have been proposed and investigated in mechanics, as well. Johnson [125] studied a steel rolling mill control system in 1972 where state-dependent delays have already been encountered. Nevertheless, state-dependent delay models have not frequently used in mechanics, since the required mathematical methods, like linearization techniques, were only recently developed (see Subsections 3.4 and 3.6).

Insperger, Stépán and Turi [119] proposed a two degree of freedom model for turning process. This models a machine tool where a workpiece is rotating, the tool cuts the surface that was formed in the previous cut, see Figure 1 below. The chip thickness is determined by the current and a previous position of the tool and the workpiece. In standard models the time delay between two succeeding cuts is considered to be equal to the period of the workpiece rotation. More realistic models given in the machine tool literature include the feed motion and the consequent trochoidal path of the cutter tooth. In this case the time delay between the succeeding cuts is not constant, it changes periodically in time. Time periodic

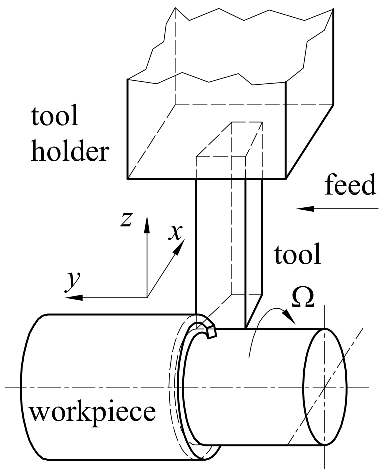

FIGURE 1. A cutting process

delays also arise in the model of varying spindle speed machining. If the regeneration process is modeled more accurately, then the vibration of the tool is also included in the model, and this results a model with state-dependent delay. The system can be modeled as a two degree of freedom oscillator that is excited by the cutting force, so the governing equations are

$$m\ddot{x}(t) + c_x\dot{x}(t) + k_x x(t) = F_x \tag{2.3.1}$$

$$m\ddot{y}(t) + c_y\dot{y}(t) + k_y y(t) = -F_y, \tag{2.3.2}$$

where the associated model parameters are mass m, damping c_x, c_y and stiffness k_x, k_y. The x and y components of the cutting force are, respectively,

$$F_x = K_x w h^q, \qquad F_y = K_y w h^q,$$

where K_x and K_y are the cutting coefficients in the x and y components, w is the depth of cut, h is the chip thickness, and the exponent $0 < q < 1$ is constant. The time delay between the present and the previous cuts is determined by the equation

$$R\Omega\tau(x_t) = 2R\pi + x(t) - x(t - \tau(x_t)), \tag{2.3.3}$$

where Ω is the spindle speed given in [rad/s] and R is the radius of the workpiece. This is an implicit equation for the time delay, and τ depends on the solution, as well. The chip thickness satisfies

$$h = v\tau(x_t) + y(t) - y(t - \tau(x_t)),$$

where v is the feed speed. Therefore the model equations are

$$\begin{aligned} m\ddot{x}(t) + c_x\dot{x}(t) + k_x x(t) &= K_x w\Big(v\tau(x_t) + y(t) - y(t - \tau(x_t))\Big)^q \\ m\ddot{y}(t) + c_y\dot{y}(t) + k_y y(t) &= -K_y w\Big(v\tau(x_t) + y(t) - y(t - \tau(x_t))\Big)^q, \end{aligned}$$

where the delay function is defined by (2.3.3). In [119] the linearized stability of an equilibrium solution was also studied using the method of [103] and [110]. It was shown that the linearized equation is almost equal to the standard constant delay machine tool vibration equation, the difference is a term with a small coefficient.

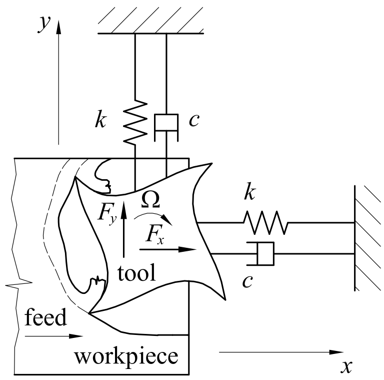

FIGURE 2. A milling process

A related problem was studied in [118], where a two degree freedom model of milling process is considered. In this machine a tool with equally spaced teeth rotating with constant spindle speed, and cuts the surface that was formed in the previous cut, see Figure 2. In this case the system again can be described by (2.3.1)–(2.3.2), but now the damping and stiffness parameters are equal in both x and y directions: $c_x = c_y = c$, $k_x = k_y = k$, and it was shown that the model equations have the form

$$(2.3.4)\ m\ddot{x}(t) + c\dot{x}(t) + kx(t) = \sum_{j=1}^{N} \alpha_{x,j}(t)\Big(R(1-\cos(\Omega\tau_j - \vartheta)) \\ +(v\tau_j + x(t-\tau_j) - x(t))\sin\varphi_j(t) + (y(t-\tau_j) - y(t))\cos\varphi_j(t)\Big)^q,$$

$$(2.3.5)\ m\ddot{y}(t) + c\dot{y}(t) + ky(t) = \sum_{j=1}^{N} \alpha_{y,j}(t)\Big(R(1-\cos(\Omega\tau_j - \vartheta)) \\ +(v\tau_j + x(t-\tau_j) - x(t))\sin\varphi_j(t) + (y(t-\tau_j) - y(t))\cos\varphi_j(t)\Big)^q.$$

Here N is the number of teeth,

$$\begin{aligned} \alpha_{x,j}(t) &= wg(\varphi_j(t))\Big(K_t\cos(\varphi_j(t)) + K_n\sin(\varphi_j(t))\Big) \\ \alpha_{y,j}(t) &= wg(\varphi_j(t))\Big(K_n\cos(\varphi_j(t)) - K_t\sin(\varphi_j(t))\Big), \end{aligned}$$

K_t and K_n are tangential and normal cutting coefficients, g is a screen function, it is equal to 1 if the j^{th} tooth is active, and it is 0 if not, $\phi_j(t) = -\Omega t + (j-1)\vartheta$, Ω is the spindle speed, $\vartheta = 2\pi/N$ is the pitch angle, and the time delays $\tau_j = \tau_j(t, x_t, y_t)$ are defined by the implicit relations

$$(2.3.6)\ \big(v\tau_j + x(t-\tau_j) - x(t)\big)\cos\phi_j(t) - \big(y(t-\tau_j) - y(t)\big)\sin\phi_j(t) = R\sin\big(\Omega\tau_j - \vartheta\big)$$

for $j = 1, \ldots, N$. It is easy to check that the functions on the right-hand-sides of (2.3.4) and (2.3.5) are periodic in time with period $\bar{\tau} = 2\pi/(N\Omega)$, and the time delay functions τ_j are periodic in time with period $T = N\bar{\tau}$. Moreover, $\tau_j(t+\bar{\tau}, x_t, y_t) = \tau_{j-1}(t, x_t, y_t)$ gives the connection between time delays associated to two succeeding cuts. Note that if the vibration of the tool is not included in the delay model, i.e., $x(t) = 0$ and $y(t) = 0$, then equation (2.3.6) is simplified to

$$v\tau_j \cos\phi_j(t) = R\sin\left(\Omega\tau_j - \vartheta\right),$$

which yields that the delay depends only on time. This case of time periodic delay was investigated earlier in the literature. If the feed is negligible relatively to the diameter, i.e., $v\tau_j \ll R$, then if we substitute $v\tau_j = 0$ into the previous equation we get $\sin(\Omega\tau_j - \vartheta) = 0$. This gives the constant delays $\tau_j = \bar{\tau} = \vartheta/\Omega$ that was usually used in standard milling models. Linearized stability of model (2.3.4)–(2.3.5) was also studied in [118] using the results of [103] and was compared to the stability of the standard time-periodic time delay models.

2.4. **Delay adaptation in neural networks and distributed systems.** A synaptic connection between two neurons is referred as to a delay line if the signal transmission is delayed. In neural circuits, time delays arise because interneural distances and axonal conduction times are finite [5, 166, 226]. In several sensory systems, delay lines are essential for coordinating activities. Examples include the auditory system of barn owls, echo location in bats, and the lateral line system of weakly electric fish. See [42] and the survey article [41]. Delay lines are also important for managing distributed systems, for such systems a fundamental problem concerns how the flow of information from distinct, independent components can be best regulated to optimize a prescribed performance of the network. For example, in parallel computing machines the asynchronous output of independent processors must be integrated to yield well-defined results.

Several possible biophysical mechanisms can be envisioned by which adjustable delays could be achieved, and recent development in the physiology of synapses and dendrites suggests that not only synaptic weights, but also synaptic delays vary [222, 1] and changing synaptic delays have significant impact on the neural signal processing.

Time delays have important influence on learning algorithms. As noted in [19], not only delays affect the learning of other parameters such as gains, time constants or synaptic weights but also *delays themselves may be part of the adjustable parameters of a neural system so as to increase the range of its dynamics.* There exist numerous examples of finely tuned delays and delay lines, and *certainly many delays are subject to variations, for instance during the growth of an organism.*

Much of the existing work related to delay adaptation in neural networks have been concentrating on the fine tuning of a selected set of parameters in architectures already endowed of a certain degree of structure. In these applications, the delays are arranged in orderly arrays of delay lines, these delay lines are essentially part of a feedforward network for which the learning task is much more simple and the delays adjust on a slow time scale. In [206], the successive parts of a spoken word are delayed differentially in order to arrive simultaneously onto a unit assigned to the recognition of that particular word. Baldi and Atiya [19] viewed this as a "time warping" technique for optimal matching in the sense that for a given input $I(t)$, the output of the ith delay line is given by the convolution

$$O_i(t) = \int_0^\infty K(i,s)I(t-s)ds,$$

where $K(i, s)$ is a Gaussian delay kernel

$$K(i, s) = \frac{1}{\sqrt{2\pi}\sigma} e^{-(s-iT)^2/2\sigma^2}$$

and T is a parameter used to discretize the possible delays. In this application, the adjustment of delays on a slow time scale may take place across several speakers during the basic development of a speech recognizer and thus the adjustment is independent of the dynamics of the network. However, it is also desirable that delays adjust on a fast scale to adapt in real time to one particular speaker during normal functioning. The above work of Baldi and Atiya, as well as those in [52, 170], develops models for delay adaptation with the help of a global teacher signal. It is shown in [52] that a network with adaptable delays can achieve smaller errors than a network with fixed delays, if they both have the same number of neurons and connections. Other rules and algorithms have been developed by which excitatory postsynaptic potentials from different synapses can be gradually pulled into coincidence, see, for example, [115].

Examples of self-organized delay adaptation can be found in [74]. For example, time delays in the optic nerve are equalized [199], signals in visual callosal axons arrive simultaneously at all axonal endings [117], and neurons in vitro can inhibit the formation of a myelin sheet by firing at a low frequency [200].

Two different mechanisms, delay shift and delay selection, are investigated in [74, 75] for the self-organized adaptation of transmission delays in the nervous system.

To formulate the model for a network consisting of a large number of presynaptic neurons and one postsynaptic neuron which receives its input lines, we consider the idealized situation where there is a continuous set of input connections described by two functions, $\rho(\tau, t)$ and $\omega(\tau, t)$, for the delays and weights, respectively: $\rho(\tau, t)d\tau$ gives the fraction of connections with delays in $[\tau, \tau + d\tau]$, and $\omega(\tau, t)$ is the average weight of connections with delay τ. We assume that delays in the system adapt on a developmental time scale, and thus the model formulated below does not involve the internal dynamics of the network. In the continuous description, the input density $J(\tau, t)$ provided by the synapses at delay τ after presentation of a pattern has the simple form

$$J(\tau, t) = \rho(\tau, t)\omega(\tau, t).$$

The dynamics of the input are governed by two simultaneous equations: a balance equation for the input density

$$\frac{\partial}{\partial t} J(\tau, t) = -\frac{\partial}{\partial \tau}[J(\tau, t)v(\tau, t)] + Q(\tau, t), \tag{2.4.1}$$

and a continuity equation for $\rho(\tau, t)$, indicating the conservation of the number of neural connections

$$\frac{\partial}{\partial} \rho(\tau, t) = -\frac{\partial}{\partial \tau}[\rho(\tau, t)v(\tau, t)]. \tag{2.4.2}$$

The drift velocity, $v(\tau, t)$, and the source term, $Q(\tau, t)$, are defined according to the Hebbian principles. In particular, in the case of delay shifts, the weights are not modified and the source term vanishes. Therefore, the dynamics is completely governed by (2.4.2) where the velocity, $v = d\tau/dt$, of the delays realizes the Hebbian adaptation

$$v(\tau,t)=\gamma_\tau\int_{-\infty}^{+\infty}W_\tau(\tau-s)P(s,t)ds, \tag{2.4.3}$$

and γ_τ denotes the learning rate, $W_\tau(x)$ denotes a learning window for delay adaptation. $W_\tau(x)$ should be positive when the presynaptic contribution precedes the postsynaptic spike, and negative in the other case, and this rule will adjust the delays such that their effects will align in time at the soma [73]. The distribution of spike times, $P(s,t)$, of a neuron depends on the input and its statistics, and this is assumed to be of the form $P(s,t)=\beta J(s,t)$ in the work of [74, 75] and it is justified if the input is sufficiently high and if there is some random background activity. Note that delay shift mechanism assumes that the transmission delays themselves are altered. This mechanism is possible because transmission velocities in the nervous system can be altered, for example, by changing the length and thickness of dendrites and axons, the extent of myelination of axons, or the density and type of ion channels. See [75, 76].

In the case of delay selection, the drift velocity of the delays vanishes and the total input of the postsynaptic neuron is not conserved. Equations (2.4.1) and (2.4.2) result in

$$\rho(\tau,t)\frac{\partial}{\partial t}\omega(\tau,t)=Q(\tau,t). \tag{2.4.4}$$

Again, the source term is derived from the Hebbian rule by

$$Q(\tau,t)=\gamma_\omega\omega(\tau,t)\rho(\tau,t)\int_{-\infty}^{+\infty}W_\omega(\tau-s)P(s,t)ds,$$

with γ_ω denoting the corresponding learning rate, and W_ω representing a learning window that is maximal just before the time of spiking–leading to a selection of delay lines for which the effects align at soma.

In the aforementioned delay-adaptation models, it is assumed that delays in the system adapt on a developmental time scale, and thus the temporal development of ρ and ω is determined by an average over an ensemble of presynaptic input patterns. In the recent work [116], the self-organized adaptation of transmission delays is incorporated into the projective adaptive resonance theory developed in [39, 40], and this self-organized adaptation of delay is driven by the dissimilarity between input patterns and stored patterns in a neural network designed for pattern recognition from data sets in high dimensional spaces. This adaptation can be regarded as a consequence of the Hebbian learning law, and the dynamic adaptation can be modeled by a nonlinear differential equation and hence a system of delay differential equations with adaptive delay is used.

We now describe the model for such a network that consists of two layers of neurons. Denote the nodes in F_1 layer (Comparison/Input Processing layer) by $P_i, i\in\Lambda_p:=\{1,\dots,m\}$; nodes in F_2 layer (Clustering layer) by $C_j, j\in\Lambda_c:=\{m+1,\dots,m+n\}$; the activation of F_1 node P_i by x_i, the activation of F_2 node C_j by y_j; the bottom-up weight from P_i to C_j by z_{ij}, the top-down weight from C_j to P_i by w_{ji}.

The STM (short-term memory) equations for neurons in F_1 layer are given by

$$\epsilon_p\frac{dx_i(t)}{dt}=-x_i(t)+I_i,\quad t\ge -1, i\in\Lambda_p, \tag{2.4.5}$$

where $0 < \epsilon_p << 1$, I_i is the constant input imposed on P_i.

The change of the STM for a F_2 neuron depends on the internal decay, the excitation from self-feedback, the inhibition from other F_2 neurons and the excitation by the bottom-up filter inputs from F_1 neurons. Namely, we have the STM equations for the committed neurons in F_2 layer:

$$\begin{aligned} \epsilon_c \tfrac{dy_j(t)}{dt} = {} & -y_j(t) + [1 - Ay_j(t)][f_c(y_j(t)) + T_j(t)] \\ & -[B + Cy_j(t)] \textstyle\sum_{k \in \Lambda_c \setminus \{j\}} f_c(y_k(t)), \ \ t \geq 0, \ \ j \in \Lambda_c, \end{aligned} \tag{2.4.6}$$

where $0 < \epsilon_c \ll 1$, $f_c : \mathbb{R} \to \mathbb{R}$ is a signal function, A, B, and C are non-negative constants, and the bottom-up filter input T_j is given by

$$T_j(t) = D \sum_{i \in \Lambda_p} z_{ij}(t) f_p(x_i(t - \tau_{ij}(t)) e^{-\alpha \tau_{ij}(t)}, \ \ t \geq 0, \tag{2.4.7}$$

where D is a scaling constant, $f_p : \mathbb{R} \to \mathbb{R}$ is the signal function of the input layer. It is assumed here that the signal transmissions between two layers are not instantaneous and the signal decays exponentially at a rate $\alpha > 0$. This assumption that signal strength decays if the transmission is delayed can be replaced by the mechanism of *delay selection* by replacing (2.4.7) by

$$T_j(t) = D_f \sum_{i \in \Lambda_p, \tau_{ij}(t) = 0} z_{ij}(t) f_p(x_i(t)) + D_d \sum_{i \in \Lambda_p, \tau_{ij}(t) > 0} z_{ij}(t) f_p(x_i(t - \tau_{ij}(t))$$

with two different weight factors $0 < D_d \ll D_f$.

The term τ_{ij} is the signal transmission delay between the cluster neuron C_j and the input neuron P_i. It is assumed that this delay is driven by the dissimilarity in the sense that the signal processing from the input neuron P_i to the cluster neuron C_j is faster when the output from P_i is similar to the corresponding component of w_{ji} of the feature vector $w_j = (w_{ji})_{i \in \Lambda_p}$ of the cluster neuron C_j. Therefore, we have

$$\beta \frac{d\tau_{ij}(t)}{dt} = -\tau_{ij}(t) + E[1 - h_{ij}(t)], \quad t \geq 0, \ \ i \in \Lambda_p, j \in \Lambda_c, \tag{2.4.8}$$

where $\beta > 0$, $E \in (0, 1)$ are constants and

$$h_{ij}(t) = S(d(f_p(x_i(t), w_{ji}(t))), z_{ij}(t))$$

is the similarity measure between the output signal $f_p(x_i(t))$ and the corresponding component $w_{ji}(t)$ of the feature vector of the cluster neuron C_j, with respect to the significance factor of the bottom-up synaptic weight $z_{ij}(t)$, here d is the usual absolute value function and $S : \mathbb{R}^+ \times [0, 1] \to [0, 1]$ is a given function, non-increasing with respect to the first argument and nondecreasing with respect to the second argument. Moreover, $S(0, 1) = 1$ (The similarity measure is 1 with complete similarity and maximal synaptic bottom-up weight) and $S(+\infty, z) = S(x, 0) = 0$ for all $z \in [0, 1]$ and $x \in \mathbb{R}^+$ (The similarity measure is 0 with complete dissimilarity or minimal bottom-up synaptic weight). Therefore, if $\tau_{ij}(0) = 0$ then from (2.4.8) it follows that $0 \leq \tau_{ij}(t) \leq E$ for all $t \in \mathbb{R}^+$, and moreover, if $h_{ij}(t) = 1$ on an interval $[0, b)$ for a given $b > 0$ then $\tau_{ij}(t) = 0$ for all $t \in [0, b)$.

The equation governing the change of the weights follows from the usual synaptic conservation rule and only connections to activated neurons are modified. The top-down weights are modified so that the template will point to the direction of the delayed and exponentially decayed outputs from F_1 layer. Therefore, we have

$$\gamma \frac{dw_{ji}(t)}{dt} = f_c(y_j(t))[-w_{ji}(t) + f_p(x_i(t-\tau_{ij}(t))e^{-\alpha\tau_{ij}(t)})] \tag{2.4.9}$$

for $t \geq 0, i \in \Lambda_p, j \in \Lambda_c$, where $\gamma > 0$ is a given constant.

The bottom-up weights are changed according to the competitive learning law and Weber Law that says that LTM size should vary inversely with input pattern scale. Thus the LTM equations for committed neurons C_j in F_2 layer are

$$\begin{aligned} \delta \tfrac{dz_{ij}(t)}{dt} = \; & f_c(y_j(t))[(1-z_{ij}(t))h_{ij}(t)L - z_{ij}(t)(1-h_{ij}(t)) \\ & -z_{ij}(t)\textstyle\sum_{k\in\Lambda_p\setminus\{i\}} h_{kj}(t)], \; t \geq 0, \; i \in \Lambda_p, j \in \Lambda_c, \end{aligned} \tag{2.4.10}$$

where $0 < \delta \ll \gamma = O(1)$ and $L > 0$ is a given constant.

Equations (2.4.5)-(2.4.10) give a system of functional differential equations where the dynamics of the delay $\tau_{ij}(t)$ is adaptive and is described by the nonlinear equation (2.4.8). The dynamics is investigated in [116] in the case where the signal functions f_p and f_c are step functions, though much remains to be done in the general case.

2.5. **Disease transmission and threshold phenomena.** Delay differential equations with state-dependent delay arise naturally from the modeling of infection disease transmission, the modeling of immune response systems and the modeling of respiration, where the delay is due to the time required to accumulate an appropriate dosage of infection or antigen concentration.

Following [208], we consider a particular infectious disease in an isolated population that is divided into several disjoint classes (compartments) of individuals: the susceptible class (those individuals who are not infective but are capable of contracting the disease and become infective), the exposed class (those who are exposed but not yet infective), and the infective class (those individuals who are capable of transmitting the disease to others), and the removed class (those who have had the disease and are dead, or have recovered and are permanently immune, or are isolated until recovery and permanent immunity occur). Let $S(t), E(t), I(t)$ and $R(t)$ be the size of each class at time t, and assume the following:

(i) the rate of exposure of susceptibles to infectives at time t is given by $-rS(t)I(t)$;
(ii) an individual who becomes infective at time t recovers from the infection (is thus removed from the population) at time $t+\sigma$, where σ is a positive constant;
(iii) an individual who is first exposed at time τ becomes infective at time t if
$$\int_\tau^t [\rho_1(x) + \rho_2(x)I(x)]dx = m,$$
where ρ_1, ρ_2 are nonnegative functions and m is a given positive constant;
(iv) The population size remains to be a constant N.

The motivation for assumption (iii), the basis for a threshold model, is that human body can often control a small exposure to an infection, that is, there is a tolerance level below which the body's immune system can combat exposure to infection. When too large an exposure results, the individual contracts the disease. The amount of exposure received depends on the duration of the exposure and the amount of infectivity around the individual, that is assumed to be proportional to the number of infective individuals in the population. Thus, during the time interval $[t, t+h]$ an exposure of $\int_t^{t+h} \rho_2(x)I(x)dx$ is accumulated where ρ_2 is a proportionality function which is a measure of the amount of infection communicated per infective (virulence). When the total exposure reaches the threshold m, the individual moves from class (E) to class (I). The term ρ_1 is the rate of accumulation of exposure independent of the number of infectives (such as constant input of virus from the external environment). In what follows, we consider the simple case where $\rho_1 = 0, \rho_2 = \rho$.

We assume the initial distribution of the infectives is given by a monotone function $I_0 : [-\sigma, 0] \to [0, \infty)$ such that $I_0(-\sigma) = 0$, and $I_0(0) > 0$ infective individuals are inserted in the population at $t = 0$. It is convenient to extend I_0 to the whole real line

$$I_0^e(t) = \begin{cases} 0, \quad |t| > \sigma; \\ I_0(t), \quad -\sigma < t \le 0; \\ I_0(0) - I_0(t-\sigma), \quad 0 \le t \le \sigma, \end{cases}$$

so that the extension $I_0^e(t)$ describes the number of initial infectives who are still present as infectives at time $t \in [-\sigma, \infty)$.

From (i) and (ii), it follows that the number of new infectives introduced into the population at time t is given by $-\int_{t-\sigma}^t \frac{d}{dx}S(\tau(x))H(x)dx$, where $H(x) = 0, x < 0$ and $H(x) = 1, x > 0$. Therefore,

$$I(t) = I_0^e(t) - \int_{t-\sigma}^t \frac{d}{dx}S(\tau(x))H(x)dx, \ \ t \ge 0.$$

For the infection to spread, some of the initial susceptible population must become infective before time σ. Thus, we assume there is $t_0 < \sigma$ so that the following "admissibility" condition

$$\int_0^{t_0} \rho(x)I_0^e(x)ds = m$$

must be met.

Then the model equations become

$$\begin{cases} \frac{d}{dt}S(t) = -rS(t)I(t), \\ I(t) = I_0^e(t) - \int_{t-\sigma}^t \frac{d}{dx}S(\tau(x))H(x)dx, \\ E(t) = S(\tau(t)) - S(t), \\ R(t) = N - S(t) - I(t) - E(t), \\ \int_{\tau(t)}^t \rho(x)I(x)dx = m, \end{cases}$$

If we further adopt the convention that $\tau(t) = 0$ for $t \le t_0$, then we obtain a state-dependent delay differential equation for $S(t)$:

$$\begin{cases} \frac{d}{dt}S(t) = -rS(t)I(t), \\ I(t) = I_0(t) + S(\tau(t-\sigma)) - S(\tau(t)), \\ \int_{\tau(t)}^t \rho(x)I(x)dx = m, t > t_0, \\ \tau(t) = 0, t \le t_0. \end{cases}$$

Other related models involving threshold conditions that determine the state-dependence of delay can be found in Gatica and Waltman [88, 89, 90], Hoppensteadt and Waltman [114], Smith [196], and Waltman [208]. Relatively complete references can be found in the work of Kuang and Smith [138, 139], where the prototype equation takes the form

$$\begin{cases} \frac{d}{dt}x(t) = -\nu x(t) - e^{-\eta\tau} f(x(t-\tau)), \\ \\ \int_{t-\tau}^{t} k(x(t), x(s))ds = m, \end{cases} \tag{2.5.1}$$

with nonnegative constants ν, η and m, and a positive function k, as well as a nonlinearity $f : \mathbb{R} \to \mathbb{R}$. Again, in the contents of epidemiological modeling, $x(t)$ may represent the proportion of a population which is infective at time t and the second equation in system (2.5.1) may reflect that an individual who is first exposed to the disease at time $t - \tau$ becomes infectious at time t if, during the interval from $t-\tau$ to t, a threshold level of exposure is accumulated where per unit time exposure depends on the infective fraction $x(s)$ via $k(x(t), x(s))$.

2.6. **Population models and size-dependent interaction.** Recent efforts in modeling state-dependent phenomena in population dynamics involve structured models, and state-dependent delay normally arises from a certain threshold condition. For example, in the work [174], Nisbet and Gurney considered insect populations which have several life stages (instars). After constructing a mathematical model consisting of an equation for the mass density function of the population, and under the homogeneity assumption for the population at each life stage, they reduced the model to a system of delay differential equations for the size of the population in each life stage. The threshold delays then appear due to the assumption that the insect must spend an amount of time in the larval stage sufficient to accumulate a threshold amount of food. See also [3].

This idea was adopted in the work of Arino, Hbid and Bravo de la Parra [8] for the growth of a population of fish where they introduce an additional stage between the eggs and the mobile larvae, the so-called (S1) larval stage. The state variable for this stage is $n_1(a,t)$, and the passage through (S1) is described with the help of another variable $q_1(a,t)$, the amount of food eaten up to time t by an individual entered in (S1) a units of time earlier. The introduction of this variable makes it possible to formulate the condition for any individual to have eaten a certain amount of food Q_1 (threshold) during the whole duration (bounded above by T_1) an individual can spend in (S1). The variation of ingested food is governed by the standard structured model (see [165]) subject to zero boundary and initial conditions that can be solved by integration along the characteristics to give

$$q_1(a,t) = \int_{t-a}^{t} \frac{K_1}{N_1(\sigma) + C_1} d\sigma, \quad t > a,$$

where K_1 is the quantity of food flowing into the species habitat per unit of volume, per unit of time, C_1 is the food(converted into a number of individuals) taken per unit of volume by consumers other than (S1) stage, and $N_1(t) = \int_0^{T_1} n_1(a,t)da$ is the population in stage (S1) which is susceptible to enter the next stage at time t per unit of volume. Hence an individual moves out of the (S1) stage exactly at time t if it entered in the (S1) stage $a_1(t)$ units of time earlier, where $a_1(t)$ is given by the threshold condition

$$\int_{t-a_1(t)}^{t} \frac{K_1}{N_1(\sigma) + C_1} d\sigma = Q_1,$$

from which it follows that

$$\frac{d}{dt}a_1(t) = -\frac{N_1(t - a_1(t)) - N_1(t)}{N_1(t) + C_1}. \tag{2.6.1}$$

The introduction of $a_1(t)$ through the above threshold condition ties the change of individual states to the dynamics of the population at the population level. From the definition of $a_1(t)$, one naturally has $n_1(a,t) = 0$ for all $a > a_1(t)$. The dynamics for $n_1(a,t)$ with $0 < a < a_1(t)$ is given by

$$\begin{cases} \frac{\partial}{\partial a}n_1(a,t) + \frac{\partial}{\partial t}n_1(a,t) = -f(a)n_1(a,t), \ 0 < a < a_1(t), t > 0, \\ n_1(a,0) = 0, \quad n_1(0,t) = B(t), \end{cases}$$

where the function f is related to the individual resistance to fluctuation of food capacities, and $B(t)$ is the density of eggs laid per unit of volume at time t. In the special case where the eggs of a given year are determined directly in terms of passive larvae that survived some years earlier, we have

$$B(t) = kN_1(t - r)$$

for some positive constants r and k. This yields

$$\begin{cases} N_1(t) = k\int_{t-a_1(t)}^{t} \exp\big(-\int_0^{t-a} f(\sigma)d\sigma\big)N_1(a-r)da, \\ \int_{t-a_1(t)}^{t} \frac{K_1}{N_1(\sigma)+C_1}d\sigma = Q_1, \end{cases} \tag{2.6.2}$$

or, equivalently,

(2.6.3)

$$\begin{cases} \frac{d}{dt}N_1(t) = kN_1(t-r) - (1 - a_1'(t))\exp\big(-\int_0^{a_1(t)} f(\sigma)d\sigma\big)kN_1(t-r) \\ \qquad -k\int_{t-a_1(t)}^{t} f(t-a)\exp\big(-\int_0^{t-a} f(\sigma)d\sigma\big)N_1(t-a)da, \\ \int_{t-a_1(t)}^{t} \frac{K_1}{N_1(\sigma)+C_1}d\sigma = Q_1. \end{cases}$$

Note that in the above system of FDEs, there are two components of the delay: a constant delay r and a state-dependent delay.

As the second equation can be written as (2.6.1) by differentiation, it is natural that Arino, Hadeler and Hbid [7] and Magal and Arino [151] considered the system with adaptive delays

$$\begin{cases} \frac{d}{dt}x(t) = -f(x(t - \tau(t))), \\ \frac{d}{dt}\tau(t) = h(x(t), \tau(t)). \end{cases}$$

The existence of periodic solutions for the above system with adaptive delays is considered, and their results will be described in Subsection 7.2.

In the context of population dynamics, the delay arises frequently as the maturation time from birth to adulthood and this time is in some cases a function of the total population [11]. In [53], the consequence of size-dependent competition among the individuals is investigated by using a system of delay differential equations with state-dependent delays for a population consisting of only two distinct size classes, juveniles and adults, under the assumption that individuals are born at a size $s = s_b$ and remain juvenile as long as $s < s_m$, and individuals mature on reaching the maturation size threshold $s = s_m$. Therefore, if the density of juvenile and adult individuals at time t are denoted by $J(t)$ and $A(t)$, respectively, and if juveniles and adults feed on a shared resource, denoted by $F(t)$, then

$$s_m - s_b = \int_{t-\tau(t)}^{t} \epsilon_g aF(x)dx, \tag{2.6.4}$$

where it is assumed that juvenile individuals feed at a rate aF and use all ingested food for growth in size with conversion efficiency ϵ_g, and $\tau(t)$ is the juvenile delay at time t. Assuming further that adult individuals feed at a rate qaF and use all ingested food for reproduction with conversion efficiency ϵ_b (here q is the ratio between the adult and juvenile feeding rate), and that the (instantaneous) mortality is inversely proportional to food intake with proportionality constant μ, de Roos and Persson obtain the following set of equations for the dynamics of $(J(t), A(t), F(t))$:

$$\left\{\begin{array}{l} \frac{dF(t)}{dt} = D - aFA - qaFA, \\ \frac{dJ(t)}{dt} = R(t) - R(t-\tau(t))P(t)\frac{F(t)}{F(t-\tau(t)))} - \frac{\mu}{aF(t)}J(t), \\ \frac{dA(t)}{dt} = R(t-\tau(t))P(t)\frac{F(t)}{F(t-\tau(t))} - \frac{\mu}{qaF(t)}A(t), \end{array}\right. \tag{2.6.5}$$

where

$$R(t) = \epsilon_b qaF(t)A(t)$$

is the total population birth rate at time t and

$$P(t) = \exp\big(-\int_{t-\tau(t)}^{t} \frac{\mu}{aF(x)}dx\big)$$

denotes the probability that an individual which should mature at time t has survived its juvenile period. The ratio $F(t)/F(t-\tau(t))$ counts for the change in maturation rate due to a change in the juvenile delay $\tau(t)$, and this can be derived by considering the model formulation in terms of a hyperbolic partial differential equation for the size distribution of the consumer population $n(t,s)$. The corresponding boundary conditions reflect the fact that the flow rate at time t across the boundary of the size domain $s = s_b$ into the juvenile class equals to total population birth rate $R(t)$ at that time, and that the flow rate at time t across the boundary $s = s_m$ into the adult class equals $\epsilon_g aF(t)n(t, s_m)$.

Note that differentiation of (2.6.4) then yields an equation to govern the evolution of the delay

$$\frac{d\tau}{dt} = 1 - \frac{F(t)}{F(t-\tau))}.$$

Similarly, in [23], it is showed that size dependent birth processes where the lifespan of individuals is a function of the current population size lead to a certain integral equation,

$$x(t) = \int_{t-L(x(t))}^{t} b(x(s))ds, \tag{2.6.6}$$

differentiation then leads to the delay differential equation

$$\frac{d}{dt}x(t) = \frac{b(x(t)) - b(x(t-L(x(t))))}{1 - L'(x(t))b(x(t-L(x(t))))}. \tag{2.6.7}$$

One should emphasize that, despite a close correspondence between solutions of (2.6.6) and (2.6.7), caution must be employed in using one equation to investigate the other. For example, any constant is a solution of (2.6.7), whereas equation (2.6.6) only admits constant solutions whose values satisfy $x = b(x)L(x)$.

We note that early development of state-dependent delay models in economics and population biology was motivated by phenomenological considerations. For example, in [25, 151] the dynamics of price adjustment in a single commodity market that involves time delays due to production lags and storage policies is considered. The importance of the incorporation of a variable production delay is pointed out as certain commodities, once produced, may be stored for a variable period of time until market prices are deemed advantageous by the producer. In the field of population dynamics, motivated by the observation in Gambell [86] that for antarctic whale and seal populations, the length of time to maturity is a function of the amount of food (mostly krill) available, Aiello, Freedman and Wu [2] propose a stage-structured model of population growth, where the time to maturity is itself state dependent and the special form is suggested by the work [6] that describes how the duration of larval development of flies is a nonlinear increasing function of larval density. This model is further analyzed in [231]. An alternative version, which is designed to address the drawback in the Aiello-Freedman-Wu model that the maturation time for any newborn depends on the existing population size at the same time, is proposed in [77] based on the assumption that the maturation time depends on the size of the population which existed at the time of birth. In particular, if $r(t)$ is the date of birth of individuals who become mature at time $t \geq 0$, then the age length up to maturity at time t will be given by the function $\tau(z(r(t)))$. Consequently, $r(t)$ can be solved implicitly by

$$r(t) = t - \tau(z(r(t))), \quad t \geq 0.$$

This modification in defining the density-delay dependent term results in the following system for the population with two stages: immature and mature, whose densities are denoted by $z(t) - x_m(t)$ and $x_m(t)$:

$$\begin{cases} \frac{d}{dt}z(t) = -\gamma z(t) + (\alpha+\gamma)x_m(t) - f(x_m(t))x_m(t), \\ \frac{d}{dt}x_m(t) = \alpha x_m(r(t))r'(t)\exp[-\gamma\tau(z(r(t)))] - f(x_m(t))x_m(t), \end{cases}$$

where $\gamma > 0$ and $f(x_m)$ are the mortality rates during the immature and mature stages. A detailed derivation of the above model and a careful analysis of conditions which assure existence, uniqueness, positiveness and boundedness of solutions can be found in [77], and the additional analysis regarding the existence of steady-state solutions and how the delay affects stability can be found in [11].

Not much progress has been made for population dynamics involving both spatial dispersal and state-dependent delay, which would naturally involve certain types of partial functional differential equations. In [190], a new class of non-local partial functional differential equations is proposed for the evolution of a single species population that involves delayed feedback, where the delay such as the time length for reproduction, is selective and the selection depends on the status of the system. The abstract model in the work [190] is the following non-local partial differential equation with state-dependent selective delay:

$$\frac{\partial}{\partial t}u(t,x) + Au(t,x) + du(t,x) = \big(F(u_t)\big)(x), x \in \Omega, \tag{2.6.8}$$

where

$$\big(F(u_t)\big)(x) := \int_{-r}^{0} \left\{ \int_{\Omega} b(u(t+\theta,y))f(x-y)dy \right\} \xi(\theta, \|u(t)\|)d\theta,$$

A is a densely-defined self-adjoint positive linear operator with domain $D(A) \subset L^2(\Omega)$ and with compact resolvent, Ω is a smooth bounded domain in $\mathbb{R}^n$, $f : \Omega - \Omega \to \mathbb{R}$ is a certain bounded function, $b : \mathbb{R} \to \mathbb{R}$ is a locally Lipschitz map and satisfies $|b(w)| \leq C_1|w| + C_2$ with $C_1 \geq 0$ and $C_2 \geq 0$, d is a positive constant. The function $u(\cdot,\cdot) : [-r,+\infty) \times \Omega \to \mathbb{R}$ is given such that for any t the function

$u(t) \equiv u(t,\cdot) \in L^2(\Omega)$, $\|\cdot\|$ is the $L^2(\Omega)$-norm. The function $\xi : [-r,0]\times\mathbb{R} \to \mathbb{R}$ represents the state-selective delay, for example, if

$$\xi(\theta,s) = e^{-\beta(g(s)-\theta)^2}, \quad \theta \in [-r,0], \quad s \in \mathbb{R},$$

then the function g gives the coordinate of the maximum of ξ. Thus the system selects the maximal historical impact on the current change rate according to the system's current state.

2.7. **Cell biology and hematological disorders.** The monograph [165] contains a brief discussion of a modified structured model for the control of the bone marrow stem cell population which supplies the circulating red blood cell population. For this case, the maturing stem cell population is structured by a maturation variable and the rate of maturation is assumed to depend only on the total mature red blood cell population. Since a threshold level of maturation is required in order for an immature cell to enter the population, a threshold delay differential equation arises naturally.

Age-structured models have also been used in the study of hematological disorders, and these models can be reduced via the method of characteristics to systems of threshold type differential delay equations, see [191, 24, 152]. Here we describe the work [152].

We first note that the precursor cells begin from a pool of burst-forming units of the erythroid line (BFU) that have differentiated into a self-sustaining population which eventually leads to the production of mature erythrocytes. At some point, the BFUs further differentiate and start down a proliferative path that can ultimately produce erythrocytes. Early in this proliferative phase of development the hormone Epo, alone with other hormones, affects the number of BFU that become erythrocytes. Increase in the concentration of Epo may increase the number of BFU recruited to mature into erythrocytes. Alternatively, there may be a relatively constant supply of committed BFU, but only the cells tagged with sufficient Epo survive the rapidly proliferating colony forming units (CFU)phase to complete maturation.

Let $p(t,\mu)$ denote the population of precursor cells at time t with age μ, and let $V(E)$ be the velocity of maturation, which may depend on the hormone concentration, E. The maturity level μ for erythropoiesis can represent the accumulation of hemoglobin in the precursor cells. If $S_0(E)$ is the number of cells recruited into the proliferating precursor population, then the entry of new precursor cells into the age-structured model satisfies the boundary condition

$$V(E)p(t,0) = S_0(E). \tag{2.7.1}$$

Let μ_F be the maximum age for a cell reaching maturity, then the dynamics of the precursor cells is governed by the age-structured model

$$\Big(\frac{\partial}{\partial t} + V(E)\frac{\partial}{\partial \mu}\Big)p(t,\mu) = [\beta(\mu,E) - H(\mu)]p(t,\mu), \quad t>0, 0<\mu<\mu_F, \tag{2.7.2}$$

where $\beta(\mu,E)$ is the net birth rate for proliferating precursor cells, and $H(\mu)$ is the disappearance rate given by

$$H(\mu) = \frac{h(\mu-\bar\mu)}{\int_\mu^{\mu_F} h(s-\bar\mu)ds}$$

and $h(\mu - \bar{\mu})$ is the distribution of maturity levels of the cells when released into the circulating blood, with $\bar{\mu}$ being the mean age of mature precursor cells and the normalization $\int_0^{\mu_F} h(\mu - \bar{\mu})d\mu = 1$.

Let $m(t,\nu)$ be the population of mature non-proliferating cells at time t and age ν, and assume the mature cells age at a constant rate W. The boundary condition for cells entering the maturation population is given by

$$Wm(t,0) = V(E)\int_0^{\mu_F} h(\mu - \bar{\mu})p(t,\mu)d\mu. \tag{2.7.3}$$

The complete feedback involves the Epo level E, its growth is regulated by the total population of mature cells. Therefore, we need to determine the maximum age $\nu_F(t)$ of erythrocytes, which varies in t as the destruction of erythrocytes occurs by active removal of the oldest cells. We assume a constant erythrocyte removal rate Q, which can be justified by either assuming a constant supply of markers or a constant number of phagocytes that become satiated in their destruction of the oldest erythrocytes, then consideration of the moving boundary condition at $\nu = \nu_F(t)$ yields

$$\Big(W - \nu_F'(t)\Big)m(t,\nu_F(t)) = Q, \tag{2.7.4}$$

or equivalently,

$$\frac{d}{dt}\nu_F(t) = W - \frac{Q}{m(t,\nu_F(t))}. \tag{2.7.5}$$

To obtain this unusual moving boundary condition, we notice that when erythrocytes age, their cell membrane breaks down and marcophages destroy the least pliable cells. On the other hand, since we assume the macrophages are in constant supply and are saturated in their consumption of erythrocytes, the age of destruction of erythrocytes then varies. In particular, during the time interval $[t, t+\Delta t]$, one can use the Mean Value Theorem to find $\chi, \eta \in [t, t+\Delta t]$ so that

$$Q\Delta t + \big[\nu_F(t+\Delta t) - \nu_F(t)\big]m(\eta,\nu_F(\eta)) = W\Delta t m(\chi,\nu_F(\chi))$$

from which (2.7.4) follows.

The dynamics of matured cells before its maximum age is governed by the age-structured model

$$\Big(\frac{\partial}{\partial t} + W\frac{\partial}{\partial \nu}\Big)m(t,\nu) = -W\gamma(\nu)m(t,\nu), \quad t > 0, 0 < \nu < \nu_F(t), \tag{2.7.6}$$

where $\gamma(\nu)$ is the age-dependent death rate of mature cells.

To complete the feedback cycle, we note that the Epo level E is governed by a negative feedback of the total population of mature cells

$$M(t) = \int_0^{\nu_F(t)} m(t,\nu)d\nu, \tag{2.7.7}$$

via

$$\frac{d}{dt}E(t) = F(M(t)) - kE(t), \tag{2.7.8}$$

where k is the decay constant for the hormone and $F(M)$ is a monotonically decreasing function of M.

In the simple case, that we consider in the remaining part of this subsection, where $V(E) = W = 1$ and $\beta(\mu, E) = \beta(\mu)$ is independent of E, if the initial condition

$$p(0,\mu) = \phi(\mu), \quad m(0,\nu) = \psi(\nu)$$

is given, then the method of characteristics yields that

$$p(t,\mu) = \begin{cases} \phi(\mu - t)\exp\Big[\int_0^t F(\mu - t + w)dw\Big], & t < \mu; \\ S_0(E(t-\mu))\exp\Big[\int_{t-\mu}^t F(w - t + \mu)dw\Big], & t > \mu, \end{cases} \tag{2.7.9}$$

with $F(\mu) = \beta(\mu) - H(\mu)$, and

$$m(t,\nu) = \begin{cases} \psi(\nu - t)\exp\Big[-\int_0^t(\nu + \sigma - t)d\sigma\Big], & t < \nu; \\ \int_0^{\mu_F} h(\mu - \bar{\mu})p(t-\nu,\mu)d\mu \exp\Big[-\int_0^\nu \gamma(\sigma)d\sigma\Big], & t > \nu. \end{cases} \tag{2.7.10}$$

If t is sufficiently large, then we get

$$M(t) = \int_0^{\nu_F(t)} \int_0^{\mu_F} h(\mu - \bar{\mu})p(t-\nu,\mu)d\mu \exp\Big[-\int_0^\nu \gamma(\sigma)d\sigma\Big]d\nu. \tag{2.7.11}$$

Thus, (2.7.5), (2.7.8) and (2.7.11), with p given in (2.7.9) form a complete system of integro-differential equations with the delay ν_F being adaptive.

3. A framework for the initial value problem

3.1. **Preliminaries.** For differential delay equations with state-dependent delays it is less obvious than in case of time-invariant delays on which state space IVPs are well-posed. For initial data in the familiar space $C = C([-h, 0]; \mathbb{R}^n)$ solutions to equations with state-dependent delay are in general not unique in cases where for similar equations with constant delay the IVP is well-posed. Winston [223] gave the following example: The functions defined by

$$x(t) = t + 1 \text{ and } x(t) = t + 1 - t^{3/2}$$

for small $t > 0$ both are solutions of the equation

$$x'(t) = -x(t - |x(t)|)$$

with initial values

$$x(t) = \begin{cases} -1, & \text{if } t \leq -1; \\ \frac{3}{2}(t+1)^{1/3} - 1, & \text{if } -1 < t \leq -\frac{7}{8}; \\ \frac{10}{7}t + 1, & \text{if } -\frac{7}{8} < t \leq 0. \end{cases}$$

Early results on existence, uniqueness, and continuous dependence for solutions of IVPs with state-dependent delays are due to Driver [58, 59, 60, 61, 62, 63], who studied cases of the two-body problem of electrodynamics; see also work by Driver and Norris [65], Travis [205], and Hoag and Driver [113]. The latter investigated equations with both delayed and advanced state-dependent shifted arguments. Winston [223, 225] studied uniqueness for special scalar equations, among others. Well-posed initial value problems are the basis for work on periodic solutions, notably by Nussbaum [178], Alt [4], Mallet-Paret and Nussbaum [154, 155, 158], Kuang and Smith [138, 139, 197], Mallet-Paret, Nussbaum and Paraskevopoulos [159], Arino, Hadeler and Hbid [7], Krisztin and Arino [134], Magal and Arino [151], and in [212]. Further existence and uniqueness results are due to Gatica and Waltman [89, 90] and Jackiewicz [121, 123], and to Ito and Kappel [120] in an approach to more general IVPs. The delay differential and integral equations addressed in these results belong to special classes where the state-dependent delay appears explicitly or is defined implicitly by an additional equation. Typically, the IVP is uniquely solved for initial and other data which satisfy suitable Lipschitz conditions. Manitius [163] and Brokate and Colonius [29] deal with differentiability of operators given by the right hand-side of differential equations with state-dependent delay, in the context of control theory. Section 6 below reports about work of Hartung and Turi [108] who proved differentiability of solutions with respect to parameters - including Lipschitz continuous initial data - in Sobolev spaces and related quasi-normed spaces. Louihi, Hbid, and Arino [147] consider a class of equations with state-dependent delay in the framework of nonlinear semigroup theory. Their results are used in an approach of Ouifki and Hbid [182] to periodic solutions which is described in Subsection 8.3 below.

Let us now see where the difficulty with uniqueness arises if a given differential equation with state-dependent delay is written in the general form (1.0.1). Our discussion of the uniqueness problem will naturally lead us to a manifold on which Equation (1.0.1) defines a continuous semiflow with continuously differentiable solution operators. We shall obtain continuously differentiable local stable and unstable manifolds of the semiflow at stationary points, also center manifolds for the solution operators, and a convenient Principle of Linearized Stability, among others.

We begin with an equation

$$x'(t) = g(x(t - r(x_t))), \tag{3.1.1}$$

with a given map $g : \mathbb{R}^n \to \mathbb{R}^n$ and a given delay functional $r : U \to [0, h]$, for $h > 0$, $n \in \mathbb{N}$, and $U \subset (\mathbb{R}^n)^{[-h,0]}$. Equation (3.1.1) has the form (1.0.1) for

$$f = g \circ ev \circ (id \times (-r))$$

where

$$ev : (\mathbb{R}^n)^{[-h,0]} \times [-h, 0] \to \mathbb{R}^n$$

is the evaluation map defined by

$$ev(\phi, s) = \phi(s).$$

Notice that the restriction of ev to $C \times [-h, 0]$ is not locally Lipschitz continuous: Lipschitz continuity would imply Lipschitz continuity of elements $\phi \in C$. Therefore in general f is not locally Lipschitz continuous on open subsets of C, and the familiar results on existence, uniqueness, and dependence of solutions on initial data and parameters for RFDEs from, say, [55, 100] fail.

The difficulty just mentioned disappears if C is replaced by the smaller Banach space $C^1 = C^1([-h, 0]; \mathbb{R}^n)$ since the restricted evaluation map

$$Ev : C^1 \times [-h, 0] \ni (\phi, s) \mapsto \phi(s) \in \mathbb{R}^n$$

is continuously differentiable, with

$$D_1 Ev(\phi, s)\chi = Ev(\chi, s) \quad \text{and} \quad D_2 Ev(\phi, s)1 = \phi'(s).$$

So, for $g : \mathbb{R}^n \to \mathbb{R}^n$ and $r : U \to [0, h]$, $U \subset C^1$ open, both continuously differentiable, the resulting functional

$$f = g \circ Ev \circ (id \times (-r))$$

is continuously differentiable from U to $\mathbb{R}^n$, with

$$\begin{aligned} Df(\phi)\chi &= Dg(\phi(-r(\phi))[D_1 Ev(\phi, -r(\phi))\chi - D_2 Ev(\phi, -r(\phi))Dr(\phi)\chi] \\ &= Dg(\phi(-r(\phi))[\chi(-r(\phi)) - Dr(\phi)\chi\, \phi'(-r(\phi))] \end{aligned}$$

for $\phi \in U$ and $\chi \in C^1$.

However, yet another obstacle is in the way. Suppose $U \subset C^1$ is open, $f : U \to \mathbb{R}^n$ is continuously differentiable, and the IVP

$$x'(t) = f(x_t) \text{ for } t > 0, \; x_0 = \phi, \tag{3.1.2}$$

is well-posed for $\phi \in U$. A solution $x : [-h, t_e) \to \mathbb{R}^n$, $0 < t_e \le \infty$, has segments $x_t \in U \subset C^1$, $0 \le t < t_e$. Therefore x is continuously differentiable, and the curve $[0, t_e) \ni t \mapsto x_t \in C^1$ is continuous. Continuity at $t = 0$ yields

$$\phi'(0) = x'(0) = f(x_0) = f(\phi),$$

a necessary condition on initial data which may or may not be satisfied.

In any case, the last equation suggests to consider the IVP (3.1.2) for initial data only in the closed subset

$$X_f = \{\phi \in U : \phi'(0) = f(\phi)\}$$

of $U \subset C^1$.

From now on, a solution is understood to be a continuously differentiable function $x : [-h, t_*) \to \mathbb{R}^n$, $0 < t_* \le \infty$, which satisfies $x_t \in U$ for $0 \le t < t_*$, $x_0 = \phi$, and $x'(t) = f(x_t)$ for $0 < t < t_*$.

Incidentally, notice that X_f is a nonlinear analogue of the domain

$$\{\phi \in C^1 : \phi'(0) = L\phi\}$$

of the generator of the semigroup given by the solutions to the linear IVP

$$y'(t) = Ly_t, y_0 = \phi \in C,$$

for $L : C \to \mathbb{R}^n$ linear continuous [55, 100].

3.2. **The semiflow on the solution manifold.** There is a mild smoothness condition (S) on the functional $f : U \to \mathbb{R}^n$, $U \subset C^1$ open, which is often satisfied if f represents an equation with state-dependent delay, and which implies all the desired results, namely

- **(S1):** f is continuously differentiable,
- **(S2):** each derivative $Df(\phi), \phi \in U$, extends to a linear map $D_e f(\phi) : C \to \mathbb{R}^n$, and
- **(S3):** the map
$$U \times C \ni (\phi, \chi) \mapsto D_e f(\phi)\chi \in \mathbb{R}^n$$
is continuous.

Let us see what condition (S) means for the example

$$f = g \circ Ev \circ (id \times (-r))$$

above, with $r : U \to [0, h]$, $U \subset C^1$ open, and $g : \mathbb{R}^n \to \mathbb{R}^n$ continuously differentiable. Suppose the delay functional r satisfies condition (S) (with $n = 1$). If then $D_e f$ is defined by the formula above for Df, with Dr replaced by $D_e r$, one sees that condition (S) holds for f. Notice that r satisfies (S) provided it is the restriction of a continuously differentiable map $V \to [0, h]$, $V \subset C$ open, to $U = V \cap C^1$.

Theorem 3.2.1. *Suppose $U \subset C^1$ is open, $f : U \to \mathbb{R}^n$ has property (S), and $X_f \neq \emptyset$. Then X_f is a continuously differentiable submanifold of U with codimension n, and each $\phi \in X_f$ uniquely defines a noncontinuable solution $x^\phi : [-h, t_+(\phi)) \to \mathbb{R}^n$ of the IVP (3.1.2). All segments x_t^ϕ, $0 \le t < t_+(\phi)$ and $\phi \in X_f$, belong to X_f, and the relations*

$$F(t, \phi) = x_t^\phi,\ \phi \in X_f,\ 0 \le t < t_+(\phi)$$

define a domain $\Omega \subset \mathbb{R} \times X_f$ and a continuous semiflow $F : \Omega \to X_f$. Each map

$$F(t, \cdot) : \{\phi \in X_f : (t, \phi) \in \Omega\} \to X_f$$

is continuously differentiable, and for all $(t, \phi) \in \Omega$ and $\chi \in T_\phi X_f$ we have

$$D_2 F(t, \phi)\chi = v_t^{\phi,\chi}$$

with the solution $v^{\phi,\chi} : [-h, t_+(\phi)) \to \mathbb{R}^n$ of the linear IVP

$$v'(t) = Df(F(t, \phi))v_t,\ v_0 = \chi. \tag{3.2.1}$$

At each (t, ϕ) with $\phi \in X_f$ and $h < t < t_+(\phi)$, the partial derivative $D_1 F(t, \phi)$ exists, and

$$D_1 F(t, \phi)1 = (x_t^\phi)'.$$

The restriction of F to the submanifold $\{(t, \phi) \in \Omega : h < t\}$ of $\mathbb{R} \times X_f$ is continuously differentiable.

Notice that the tangent spaces of the manifold X_f are given by

$$T_\phi X_f = \{\chi \in C^1 : \chi'(0) = Df(\phi)\chi\}.$$

Theorem 3.2.1 is proved in [213, 214].

The first part (S1) of the hypothesis and continuity of each extension $D_e f(\phi)$ suffice for X_f to be a continuously differentiable submanifold. The proof of this is simple: For

$$p : C^1 \ni \phi \mapsto \phi'(0) \in \mathbb{R}^n,$$
$$X_f = (p - f)^{-1}(0),$$

and the Implicit Function Theorem yields local representations of X_f as graphs over the kernels of the linear maps $D(p-f)(\phi) = p - Df(\phi))$, $\phi \in X_f$, provided these linear maps are surjective. In case $n = 1$ surjectivity follows since using the continuity of $D_e f(\phi)$ at $0 \in C$ one finds $\chi \in C^1 \subset C$ with $\chi'(0) = 1$ and $Df(\phi)\chi = D_e f(\phi)\chi < 1 = p\chi$, which gives $(p - Df(\phi))C^1 = \mathbb{R}$. In case $n > 1$ similar arguments yield a basis of $\mathbb{R}^n$ in $(p - Df(\phi))C^1$.

Condition (S3) implies that the map $D_e f$ is locally bounded. From this one derives easily the following local Lipschitz property:

(L): For every $\phi \in U$ there are a neighbourhood V and $L \geq 0$ with $|f(\psi) - f(\chi)| \leq L\|\psi - \chi\|_C$ for all $\psi \in V, \chi \in V$,

see Corollary 1 in [214]. Notice that the last norm is the norm on C and not the larger norm on the smaller space C^1. So (L) is not a consequence of (S1). Property (L) (together with (S1), (S2) and continuity of each $D_e f(\phi)$) yields existence and continuity of the semiflow as well as the properties of the maps $F(t, \cdot)$ stated in Theorem 3.2.1. Only for the two last statements of Theorem 3.2.1, on smoothness, the full hypothesis (S) is needed.

It is worth noting that continuity of the map

$$U \ni \phi \mapsto D_e f(\phi) \in L(C, \mathbb{R}^n),$$

which seems only slightly stronger than property (S3) above, does in general *not* hold for functionals f which represent differential equations with state-dependent delay.

A key issue in the proof of Theorem 3.2.1 is how the local Lipschitz property (L) is used for the construction of local solutions to the IVP. Let us briefly explain this.

In order to solve the integrated version

$$x(t) = \phi(0) + \int_0^t f(x_s)ds,\ x_0 = \phi \in X_f,$$

of the IVP for $0 \leq t \leq T$ by a continuously differentiable map

$$x : [-h, T] \to \mathbb{R}^n,$$

which is continuously differentiable with respect to ϕ, the desired solution is first written as the sum of a linear, continuously differentiable continuation $\hat{\phi}$ of ϕ and of a function u which is zero on $[-h, 0]$. The fixed point problem for u is

$$u(t) = \int_0^t (f(u_s + \hat{\phi}_s) - f(\phi))ds,\ 0 \leq t \leq T.$$

The advantage of this formulation is that for the operator $u \mapsto A(\phi, u)$ given by the right hand side dependence on ϕ is more explicit than in the equation for x. For $A(\phi, \cdot)$ to become a contraction with respect to the norm $\|u\|_{T,1} = \max_{0\le s\le T}|u(s)| + \max_{0\le s\le T}|u'(s)|$ we use the following estimate for $v = A(\phi, u)$, $\overline{v} = A(\phi, \overline{u})$, and $0 \le t \le T$:

$$|v'(t) - \overline{v}'(t)| = |f(u_t + \hat{\phi}_t) - f(\overline{u}_t + \hat{\phi}_t)| \le L\|u_t - \overline{u}_t\|_C$$

(due to the local Lipschitz property, for T and u and $\overline{u}$ small)

$$\le L \max_{0\le s\le T}|u(s) - \overline{u}(s)| = L \max_{0\le s\le T}|\int_0^s (u'(r) - \overline{u}'(r))dr| \le L\,T\,\|u - \overline{u}\|_{T,1}.$$

The proof that the semiflow F is continuously differentiable on the manifold given by $t > h$ is based on growth estimates and smoothness properties of solutions to the IVP

$$v'(t) = D_e f(F(t, \phi))v_t, v_0 = \chi \tag{3.2.2}$$

for initial data $\chi \in C$.

The previous result is optimal in the sense that typically the semiflow F has no partial derivatives with respect to the first variable for $0 \le t < h$.

The framework presented up to here is instrumental in the proof in [215] that for a certain differential system with state-dependent delay, which models position control by echo, hyperbolic stable periodic orbits exist.

With regard to results for solutions $x : [-h, t_+) \to \mathbb{R}^n$ to equations of the form (3.1.1) and generalizations thereof in the sense that x is continuous but differentiable only for $0 < t < t_+$, notice that for $h \le t < t_+$ we have

$$x_t \in X_f$$

(with $f = g\circ ev\circ(id\times(-r))$ in case of Equation (3.1.1)), so all dynamical properties like structure and stability of invariant sets (stationary points, periodic orbits, unstable manifolds, global attractors, ...) are determined by the semiflow F on X_f - provided the mild smoothness condition (S) holds.

Some notions of the approach to well-posedness and smoothness which we described here are related to ideas from earlier work. The Lipschitz property (L) from the proof of Theorem 3.2.1 was used before in [212] and is analogous to the notion of being *locally almost Lipschitzian* from [159]. The condition that (S1) holds and each $D_e f(\phi)$ is continuous was introduced in [159] as *almost Fréchet differentiability.* It also is a special case of a smoothness condition used in [131]. Sets analogous to X_f were considered in [136] as state space for neutral functional differential equations, and by Louihi, Hbid and Arino [147]. In [147] the IVP for a certain class of equations of the form (3.1.1) defines a semigroup of locally Lipschitz continuous solution operators on the space $C^{0,1}$. The positively invariant subset $E \subset C^{0,1}$ of data $\phi \in C^1$ with $\phi'(0) = g(\phi(-r(\phi)))$ is identified as the domain of strong continuity of the semigroup. In the present notation, $E = X_f$ for $f = g \circ Ev \circ (id \times (-r))$. It is stated in [147] that E is a Lipschitz manifold.

3.3. **Compactness.** Recall that a map from a subset of a Banach space into a Banach space is called compact if images of bounded sets have compact closure. A simple compactness result for the semiflow of Theorem 3.2.1 is the following. Notice in which way the hypotheses strengthen property (S).

Proposition 3.3.1. *Suppose $f : U \to \mathbb{R}^n$, $U \subset C^1$ open, is bounded, satisfies condition (S), and*

(Lb): *for every bounded set $B \subset U$ there exists $L_B \geq 0$ with $|f(\phi) - f(\psi)| \leq L_B\|\phi - \psi\|_C$ for all ϕ, ψ in B.*

Then all maps $F(t, \cdot)$, $t \geq h$, are compact.

Proof. Let $t \geq h$ and let a bounded set $B \subset \{\phi \in X_f : (t, \phi) \in \Omega\}$ be given. In order that $\overline{F(t, B)} \subset C^1$ be compact it is enough to show that every sequence in $F(t, B)$ has a convergent subsequence, which follows by means of the Ascoli-Arzelà Theorem provided both sets $M = \{x_t^\phi \in C^1 : \phi \in B\}$ and $M' = \{(x_t^\phi)' \in C : \phi \in B\}$ are bounded with respect to the norm $\| \cdot \|_C$ and equicontinuous. The boundedness of B and f in combination with Equation (1.0.1) shows that

$$b = \sup_{\phi \in B, -h \leq s \leq t} |(x^\phi)'(s)| < \infty.$$

Then the boundedness of B and integration yield

$$\sup_{\phi \in B, -h \leq s \leq t} |x^\phi(s)| < \infty.$$

In particular, M' and M are bounded with respect to $\| \cdot \|_C$. The boundedness of M' yields the equicontinuity of M. We also have that the set

$$Y = \{F(s, \phi) : \phi \in B, 0 \leq s \leq t\} \subset X_f \subset U$$

is bounded with respect to $\| \cdot \|_{C^1}$. Now equicontinuity of M' follows from the estimate

$$\begin{aligned}
|(x_t^\phi)'(s) - (x_t^\phi)'(u)| = |(x^\phi)'(t+s) - (x^\phi)'(t+u)| &= |f(F(t+s, \phi)) - f(F(t+u, \phi))| \\
&\leq L_Y \|F(t+s, \phi) - F(t+u, \phi)\|_C \\
&= L_Y \max_{-h \leq v \leq 0} |x^\phi(t+s+v) - x^\phi(t+u+v)| \\
&= L_Y \max_{-h \leq v \leq 0} \left| \int_{t+u+v}^{t+s+v} (x^\phi)'(w) dw \right| \leq L_Y\, b\, |s-u|
\end{aligned}$$

for all $\phi \in B$ and all s, u in $[-h, 0]$. ☐

In case of the example $f = g \circ Ev \circ (id \times (-r))$ with $r : U \to [0, h]$, $U \subset C^1$ open, and $g : \mathbb{R}^n \to \mathbb{R}^n$ continuously differentiable the hypotheses of Proposition 3.3.1 are satisfied if in addition U is convex, r has property (S), g is bounded, and $D_e r$ and Dg map bounded sets onto bounded sets: The boundedness of f is obvious. The formula which computes $D_e f$ from Dg and $D_e r$ shows that $D_e f$ maps bounded sets onto bounded subsets. For a given bounded set $B \subset U$ there is a larger convex bounded subset $B^* \subset U$, and for ϕ, ψ in B^* we have

$$\begin{aligned}
|f(\phi) - f(\psi)| &= \left| \int_0^1 Df(\psi + t(\phi - \psi))(\phi - \psi) dt \right| \\
&= \left| \int_0^1 D_e f(\psi + t(\phi - \psi))(\phi - \psi) dt \right| \\
&\leq \int_0^1 |D_e f(\psi + t(\phi - \psi))(\phi - \psi)| dt \\
&\leq \sup_{\chi \in B^*} \|D_e f(\chi)\| \|\phi - \psi\|_C,
\end{aligned}$$

which yields property (Lb).

For another result on compactness, for solution operators on the space $C^{0,1}$, see [147].

3.4. **Linearization at equilibria.** Theorem 3.2.1 reveals in particular how to linearize semiflows defined by differential equations with state-dependent delay, an issue which had been mysterious before. Consider $f : U \to \mathbb{R}^n$, $U \subset C^1$ open, with property (S). Let a stationary point $\phi_0 \in X_f$ of the semiflow F from Theorem 3.2.1 be given. The linearization of F at ϕ_0 is the semigroup T of the linear continuous operators $T(t) = D_2F(t, \phi_0)$, $t \geq 0$, on the Banach space $T_{\phi_0}X_f$ with the norm $\| \cdot \|_{C^1}$. T is strongly continuous since the solutions $v^{\phi_0,\chi} : [-h, \infty) \to \mathbb{R}^n$ of the IVP (3.2.1) with $\chi \in T_{\phi_0}X_f$ are continuously differentiable.

Before the present approach was available a heuristic technique had been developped in order to circumvent the linearization problem in studies of local stability and instability properties of equilibria. This technique associates to the given nonlinear equation an auxiliary linear equation in the following way: First the delay is frozen at equilibrium, then the resulting nonlinear equation with constant delay is linearized. Of course, this makes sense only for equations where the delay appears explicitly, like e.g. in Equation (3.1.1). Let us use Equation (3.1.1) to show that the auxiliary equation found by the heuristic technique is

$$v'(t) = D_ef(\phi_0)v_t$$

in our framework. Suppose for simplicity that $n = 1$, that $g : \mathbb{R} \to \mathbb{R}$ satisfies $g(0) = 0$ and that g and the delay functional $r : U \to [0, h]$, $U \subset C^1$ open, are continuously differentiable. Freezing the delay in Equation (3.1.1) at $\phi_0 = 0$ and linearizing the resulting RFDE

$$x'(t) = g(x(t - r(0)))$$

at the zero solution yields

$$v'(t) = g'(0)v(t - r(0)).$$

On the other hand, for the functional $f_{g,r} = g \circ Ev \circ (id \times (-r))$ and $\chi \in C^1$ the computation of derivatives above yields

$$Df_{g,r}(0)\chi = Dg(0)\chi(-r(0)) = g'(0)\chi(-r(0)).$$

Obviously the right hand side of this equation defines a continuous linear extension $D_ef_{g,r}(0) : C \to \mathbb{R}$ of $Df_{g,r}(0)$, which verifies our previous statement. In other words, the auxiliary equation found by the heuristic method yields the true linear variational equation by restriction to the tangent space T_0X_f.

In the sequel we clarify the relation between the spectral properties of the linearization T of the semiflow F of Theorem 3.2.1 at a stationary point $\phi_0 \in X_f$ and the strongly continuous semigroup T_e on the space C which is defined by the solutions of the IVP (3.2.2). Recall that the generator $G_e : dom \to C$ of T_e is given by $dom = \{\phi \in C^1 : \phi'(0) = D_ef(\phi_0)\phi\}$ and $G_e\phi = \phi'$.

We have $T_{\phi_0}X_f = dom$, $T(t)\phi = T_e(t)\phi$ for $t \geq 0$ and $\phi \in dom$, and the norm $\| \cdot \|_{C^1}$ coincides with the graph norm $\| \cdot \|_e = \| \cdot \|_C + \|G_e \cdot \|_C$ of the operator G_e.

It is a simple general fact that for a strongly continuous semigroup S of linear continuous operators on a Banach space B, with generator $A : D_A \to B$, the induced linear operators

$$D_A \ni x \mapsto S(t)x \in D_A$$

are continuous with respect to the graph norm $\| \cdot \|_A = \| \cdot \| + \|A \cdot \|$ and form a strongly continuous semigroup on the Banach space $(D_A, \| \cdot \|_A)$, with the generator

A_d defined on the domain
$$D_d = \{x \in D_A : Ax \in D_A\}$$
of A^2 and given by $A_d x = Ax$. Proofs are immediate from $S(t)D_A \subset D_A$ and
$$S(t)Ax = AS(t)x \text{ on } D_A$$
for all $t \geq 0$.

Our semigroups and their generators are precisely in the relation just described, so we have
$$\begin{aligned} D = (D_d =)&\{\chi \in C^1 : \chi'(0) = Df(\phi_0)\chi, \chi' \in C^1, \chi''(0) = Df(\phi_0)\chi'\} \\ &= \{\chi \in C^2 : \chi'(0) = Df(\phi_0)\chi, \chi''(0) = Df(\phi_0)\chi'\} \end{aligned}$$
and $G\chi = \chi'$ on D.

We return to the general case. Below, ρ and σ denote resolvent sets and spectra, respectively.

Proposition 3.4.1. *Suppose B is a Banach space over $\mathbb{C}$, S is a strongly continuous semigroup on B with generator $A : D_A \to B$, and A_d is the generator of the semigroup induced on the Banach space $(D_A, \|\cdot\|_A)$. Then the following holds.*

- **(i):** *$\rho(A) \subset \rho(A_d) \subset \rho(A) \cup \{\lambda \in \mathbb{C} : A - \lambda I$ injective and $(A - \lambda I)D_A \neq B\}$, and $\sigma(A_d) \subset \sigma(A)$. For all $\lambda \in \rho(A)$, $(A_d - \lambda I)^{-1}x = (A - \lambda I)^{-1}x$ on D_A.*
- **(ii):** *Let γ be a simple closed curve in $\rho(A)$and let $\sigma_{\gamma,d} = int(\gamma) \cap \sigma(A_d)$, $\sigma_\gamma = int(\gamma) \cap \sigma(A)$. Then the spectral projections P_d, P and generalized eigenspaces $\mathcal{G}_d = P_d D_A$, $\mathcal{G} = PB$ associated with the sets $\sigma_{\gamma,d} \subset \sigma(A_d)$ and $\sigma_\gamma \subset \sigma(A)$, respectively, satisfy*
$$P_d x = Px \text{ on } D_A,\ \mathcal{G}_d \subset \mathcal{G},\ \text{and } \overline{\mathcal{G}_d}^B = \mathcal{G}.$$

Proof. 1. Proof of (i).

1.1. Let $\lambda \in \rho(A)$ be given. The continuity of the inverse with respect to the norm on B and the definition of the graph norm combined show that for all $y \in D_A$ we have
$$\begin{aligned} ||(A - \lambda I)^{-1}y||_A &= ||(A - \lambda I)^{-1}y|| + ||A(A - \lambda I)^{-1}y|| \\ &\leq ||(A - \lambda I)^{-1}y|| + ||(A - \lambda I)(A - \lambda I)^{-1}y|| + ||\lambda(A - \lambda I)^{-1}y|| \\ &\leq ((1 + |\lambda|)||(A - \lambda I)^{-1}|| + 1)||y||. \end{aligned}$$
It follows that $(A - \lambda I)^{-1}$ defines a linear continuous map R from $(D_A, \|\cdot\|_A)$ into itself. The range of R contains the domain D_d of the generator A_d since for any $x \in D_A$ with $Ax \in D_A$ we have that $y = (A_d - \lambda I)x = (A - \lambda I)x \in D_A$, hence $x = (A - \lambda I)^{-1}y \in (A - \lambda I)^{-1}D_A = RD_A$.

Conversely, for $x = Ry = (A - \lambda I)^{-1}y$ with $y \in D_A$ we have $x \in D_A$ and $(A - \lambda I)x = y \in D_A$, hence $Ax = \lambda x + y \in D_A$, or $x \in D_d$.

So R maps D_A injectively onto the domain D_d of A_d. We have $R(A_d - \lambda I)y = R(A - \lambda I)y = y$ on D_d, $(A_d - \lambda I)Rx = (A - \lambda I)Rx = x$ on D_A, and it follows that $\lambda \in \rho(A_d)$ and $(A_d - \lambda I)^{-1}y = (A - \lambda I)^{-1}y$ on D_A.

The shown inclusion of resolvent sets yields the asserted result for spectra.

1.2. Proof of the remaining inclusion. Let $\lambda \in \rho(A_d)$ be given. $A - \lambda I$ is injective since $(A - \lambda I)x = 0$ implies $Ax \in D_A$, or $x \in D_d$, and therefore $(A_d - \lambda I)x = (A - \lambda I)x = 0$, which in turn gives $x = 0$, by $\lambda \in \rho(A_d)$.

Consider the injective continuous map
$$A_c : (D_A, \|\cdot\|_A) \ni x \mapsto (A - \lambda I)x \in B.$$

In case $(A - \lambda I)D_A = B$ the open mapping theorem shows that the inverse of A_c is continuous, which implies that

$$B \ni y \mapsto (A - \lambda I)^{-1} y \in B$$

is continuous. So in this case, $\lambda \in \rho(A)$, and the assertion becomes obvious.

2. Proof of (ii). The representation of $P_d x$ and Px, $x \in D_A$, as contour integrals in $(D_A, \|\cdot\|_A)$ and B, respectively, along the curve γ with integrands given by

$$(A_d - \lambda I)^{-1} x = (A - \lambda I)^{-1} x, \ \lambda \in |\gamma|,$$

shows that $P_d x = Px$ on D_A. Consequently,

$$\mathcal{G}_d = P_d D_A \subset P D_A \subset PB = \mathcal{G},$$

and finally

$$\overline{\mathcal{G}_d}^B = \overline{P_d D_A}^B = \overline{P D_A}^B = PB$$

(since $\overline{D_A} = B$ and since P has closed range)

$$= \mathcal{G}.$$

□

It is easy to see that for a given semiflow S on a Banach space B over $\mathbb{R}$ and its generator $A : D_A \to B$, the complexification $A_{\mathbb{C}}$ of A coincides with the generator of the complexified semigroup $S_{\mathbb{C}} : t \mapsto S(t)_{\mathbb{C}}$, and that the semigroup induced by $S_{\mathbb{C}}$ on $(D_A)_{\mathbb{C}}$ is generated by the complexification of the generator $A_d : D_d \to (D_A, \|\cdot\|_A)$.

Returning to our case of semigroups T_e on C, T on $T_{\phi_0} X_f$ and their generators G_e, G, respectively, we recall that the embedding $(dom, \|\cdot\|_e) \to C$ is compact, by the Ascoli-Arzelà theorem. This yields that all resolvents of $(G_e)_{\mathbb{C}}$, which define continuous maps from $C_{\mathbb{C}}$ onto the complexification of the Banach space $(dom, \|\cdot\|_e)$, are compact. Therefore $\sigma(G_e) := \sigma((G_e)_{\mathbb{C}})$ is discrete and consists of eigenvalues with finite-dimensional generalized eigenspaces. Using Proposition 3.4.1 and the remarks following it we infer

$$\rho(G) = \rho(G_e), \sigma(G) = \sigma(G_e),$$

and for the spectral projections and generalized eigenspaces of $G_{\mathbb{C}}$ and $(G_e)_{\mathbb{C}}$ which are associated with $\lambda \in \sigma(G) = \sigma(G_e)$,

$$P(\lambda)\chi = P_e(\lambda)\chi \text{ on } T_{\phi_0} X_f = dom,$$

$$\mathcal{G}(\lambda) = \mathcal{G}_e(\lambda).$$

Recall that to the right of any line parallel to the imaginary axis there are at most a finite number of eigenvalues of G_e. Let C_u and C_c denote the unstable and center spaces of G_e, i.e., the finite-dimensional realified generalized eigenspaces given by the eigenvalues with positive real part and on the imaginary axis, respectively. The stable space C_s of G_e is the realified generalized eigenspace given by the spectrum with negative real part. More precisely, C_s is the realification of the space $(id - P_{\geq 0})C_{\mathbb{C}}$ where $P_{\geq 0} : C_{\mathbb{C}} \to \mathbb{C}_{\mathbb{C}}$ denotes the spectral projection associated with the finite spectral set of all $\lambda \in \sigma(G_e)$ with $Re\lambda \geq 0$. We have

$$C_u \oplus C_c \subset D \subset dom = T_{\phi_0} X_f$$

while the infinite-dimensional, complementary space C_s is not contained in dom. Using the previous relations between spectral projections and generalized eigenspaces of the generators G and G_e it follows easily that the unstable and center spaces of G coincide with C_u and C_c, respectively, and that the stable space of G is

$$C_s \cap dom.$$

For each map $F(t,\cdot)$, $t > 0$, we have

$$\sigma(D\,F(t,\cdot)(\phi_0)) = \sigma(T(t)) \subset \{0\} \cup \{e^{z\,t} : z \in \sigma(G)\}, \tag{3.4.1}$$

and the linear unstable, center, and stable spaces for the derivative $D\,F(t,\cdot)(\phi_0) = D_2F(t,\phi_0) = T(t)$, which are defined by the eigenvalues outside, on, and inside the unit circle, are

$$C_u,\ C_c,\ \text{and}\, C_s \cap T_{\phi_0}X_f,\ \text{respectively.}$$

3.5. **Local invariant manifolds at stationary points.** Consider $f : U \to \mathbb{R}^n$, $U \subset C^1$ open, with property (S) and the associated semiflow $F : \Omega \to X_f$ of Theorem 3.2.1. It is convenient to set

$$\Omega_t = \{\phi \in X_f : (t,\phi) \in \Omega\} \text{ and } F_t = F(t,\cdot),$$

for each $t \geq 0$. A stationary point $\phi_0 \in X_f$ of the semiflow F is a fixed point for each map $F_t : \Omega_t \to X_f$, $t > 0$. Results about local invariant manifolds for continuously differentiable maps on open subsets of Banach spaces [99, 45, 175, 135] yield continuously differentiable local stable, center, and unstable manifolds

$$W_s \subset X_f,\ W_c \subset X_f,\ W_u \subset X_f$$

of F_t $t > 0$, at the fixed point ϕ_0, with tangent spaces at ϕ_0 given by

$$T_{\phi_0}W_s = C_s \cap T_{\phi_0}X_f,\ T_{\phi_0}W_c = C_c,\ T_{\phi_0}W_u = C_u.$$

In the sequel we recall details of this and use them to show that local stable manifolds W_s of any map F_a, $a > 0$, provide local stable manifolds of the semiflow F at the given stationary point. The last fact is familiar for flows of continuously differentiable vectorfields in finite dimensions, but here we are concerned with semiflows on a Banach manifold of infinite dimension, and some care should be taken which properties of the underlying delay differential equation (1.0.1) ensure the result.

For local unstable manifolds an analogous result holds, and the proof is similar. Existence of smooth local center manifolds for the semiflow is more difficult, and the relationship between center manifolds of its time-t-maps and center manifolds for the semiflow is more subtle. This will be discussed in the next section.

So let a stationary point $\phi_0 \in X_f$ of the semiflow F be given. Recall from the preceding subsection the linear stable, center, and unstable spaces

$$C_s \cap T_{\phi_0}X_f,\ C_c,\ C_u$$

of the generator G of the semigroup T on $T_{\phi_0}X_f$. Choose $\beta > 0$ so that

$$-\beta > \max\{Re\,z : z \in \sigma(G),\ Re\,z < 0\}.$$

Let $a > 0$. In order to introduce the local stable manifold of the map $F_a : \Omega_a \to X_f$ at the fixed point ϕ_0 we use a manifold chart of X_f at ϕ_0. Let $Y = T_{\phi_0}X_f$. There is a subspace $E \subset C^1$ of dimension n which is a complement of Y in C^1. Let $P : C^1 \to C^1$ denote the projection along E onto Y. Then the equation $K(\phi) = P(\phi - \phi_0)$ defines a manifold chart on an open neighbourhood V of ϕ_0 in $\Omega_a \subset X_f$, with $Y_0 = K(V)$ an open neighbourhood of $0 = K(\phi_0)$ in the Banach space Y (with the norm given by $\|\cdot\|_{C^1}$). The inverse of K is given by a continuously differentiable map $R : Y_0 \to C^1$. Both derivatives $DK(\phi_0)$ and $DR(0)$ are given by the identity on Y. We may assume that there is a Lipschitz constant $L_R \geq 0$ so that

$$\|R(\chi) - R(\psi)\|_{C^1} \leq L_R \|\chi - \psi\|_{C^1} \text{ for all } \chi \in Y_0,\ \psi \in Y_0.$$

Choose an open neighbourhood $Y_1 \subset Y_0$ of 0 in Y with $F_a(R(Y_1)) \subset V$. In local coordinates the map F_a is represented by the continuously differentiable map

$$H : Y_1 \ni \chi \mapsto K(F_a(R(\chi))) \in Y.$$

Obviously, $H(0) = 0$, $DH(0) = DF_a(\phi_0) = T(a)$, and $H(Y_1) \subset Y_0$. Using the last statement of the preceding subsection we infer that the linear stable, center, and unstable spaces of H at its fixed point $0 \in Y_1$ are $C_s \cap Y$, C_c, and C_u, respectively. Set $\lambda = e^{-a\,\beta}$. Then (3.4.1) gives

$$\max\{|\zeta| : \zeta \in \sigma(DH(0)), |\zeta| < 1\} < \lambda < 1.$$

The Stable Manifold Theorem (see Theorem I.2 in [135]) yields the following result.

Proposition 3.5.1. *There exist $\alpha \in (0, \lambda)$, convex open neighbourhoods $C_{s,2}$ of 0 in $C_s \cap Y$ and $C_{cu,2}$ of 0 in $C_c \oplus C_u$ with $N = C_{s,2} + C_{cu,2} \subset Y_2$, a continuously differentiable map $w : C_{s,2} \to C_{cu,2}$ with $w(0) = 0$ and $Dw(0) = 0$, and an equivalent norm $\|\cdot\|_H$ on Y such that the following holds.*

- **(i):** *The graph $W = \{\chi + w(\chi) : \chi \in C_{s,2}\}$ is equal to the set of all initial points $\psi = \psi_0$ of trajectories $(\psi_j)_0^\infty$ of H which satisfy $\lambda^{-j}\psi_j \in N$ for all $j \in \mathbb{N}_0$ and $\lambda^{-j}\psi_j \to 0$ as $j \to \infty$.*
- **(ii):** *$H(W) \subset W$.*
- **(iii):** *$\|H(\phi) - H(\psi)\|_H \leq \alpha\|\phi - \psi\|_H$ for all $\psi \in W$, $\phi \in W$.*
- **(iv):** *For every trajectory $(\psi_j)_0^\infty$ of H with $\lambda^{-j}\psi_j \in N$ for all $j \in \mathbb{N}_0$,*

$$\psi_0 \in W.$$

Here, trajectories are defined by the equations $\psi_{j+1} = H(\psi_j)$ for all integers $j \geq 0$.

The local stable manifold of F_a at ϕ_0 is the continuously differentiable submanifold

$$W_s = R(W)$$

of X_f. Obviously, $W_s \subset V$, $\phi_0 \in W_s$, and $T_{\phi_0}W_s = C_s \cap Y = C_s \cap T_{\phi_0}X_f$.

Corollary 3.5.2. **(i):** *$F_a(W_s) \subset W_s$, and each neighbourhood of ϕ_0 in W_s contains a neighbourhood $W_{s,1}$ of ϕ_0 in W_s with $F_a(W_{s,1}) \subset W_{s,1}$.*

- **(ii):** *There exists $c_s \geq 0$ so that for every trajectory $(\psi_j)_0^\infty$ of F_a in W_s and for all integers $j \geq 0$,*

$$\|\psi_j - \phi_0\|_{C^1} \leq c_s\alpha^j\|\psi_0 - \phi_0\|_{C^1}.$$

Proof. 1. The first inclusion in assertion (i) follows from

$$K(F_a(W_s)) = K(F_a(R(W))) = H(W) \subset W = K(R(W))$$

by application of R. Proof of the second part of (i): For $\epsilon > 0$, set

$$Y_{H,\epsilon} = \{\psi \in Y : \|\psi\|_H < \epsilon\}.$$

Any given neighbourhood of ϕ_0 in $V \subset X_f$ contains $V_\epsilon = R(Y_{H,\epsilon})$ for some $\epsilon > 0$, and $R(W \cap Y_{H,\epsilon}) = R(W) \cap R(Y_{H,\epsilon}) = W_s \cap V_\epsilon$. Part (iii) of Proposition 3.5.1 yields $F_a(W_s \cap V_\epsilon) = R(K(F_a(R(W \cap Y_{H,\epsilon})))) = R(H(W \cap Y_{H,\epsilon})) \subset R(W \cap Y_{H,\epsilon}) = W_s \cap V_\epsilon$.

2. Proof of assertion (ii). There are positive constants $c_1 \le c_2$ with

$$c_1\|\chi\|_{C^1} \le \|\chi\|_H \le c_2\|\chi\|_{C^1} \text{ for all } \chi \in Y.$$

Let a trajectory $(\psi_j)_0^\infty$ of F_a in W_s be given. The points $\chi_j = K(\psi_j) \in W$ form a trajectory of H since

$$\chi_{j+1} = K(\psi_{j+1}) = K(F_a(\psi_j)) = K(F_a(R(\chi_j))) = H(\chi_j)$$

for each integer $j \ge 0$. Hence

$$\begin{aligned} \|\psi_j - \phi_0\|_{C^1} &= \|R(\chi_j) - R(0)\|_{C^1} \\ &\le L_R\|\chi_j\|_{C^1} \\ &\le L_R c_1^{-1}\|\chi_j\|_H \\ &\le L_R c_1^{-1}\alpha^j\|\chi_0\|_H \\ &\le L_R\frac{c_2}{c_1}\alpha^j\|\chi_0\|_{C^1} \\ &= L_R\frac{c_2}{c_1}\alpha^j\|P\|_{C^1}\|\psi_0 - \phi_0\|_{C^1}. \end{aligned}$$

□

We want to show that the semiflow F maps a piece W^s of W_s close to ϕ_0 into W_s, in the sense that $F([0,\infty) \times W^s) \subset W_s$. The proof requires a quantitative version of continuous dependence on initial conditions. Notice that the proof of the latter employs the Lipschitz property (L) of the functional f.

Proposition 3.5.3. *There exist an open neighbourhood $X_{f,a}$ of ϕ_0 in X_f and a constant $c_a \ge 0$ so that $[0,a] \times X_{f,a} \subset \Omega$ and*

$$\|F(t,\phi) - \phi_0\|_{C^1} \le c_a\|\phi - \phi_0\|_{C^1} \textit{ for all } (t,\phi) \in [0,a] \times X_{f,a}.$$

Proof. 1. Using continuity of the semiflow and compactness of the interval $[0,a]$ we find an open neighbourhood $X_{f,a}$ of the stationary point ϕ_0 in X_f so that $[0,a] \times X_{f,a} \subset \Omega$ and $F([0,a] \times X_{f,a})$ is contained in a neighbourhood of ϕ_0 in the domain U of f on which the Lipschitz estimate (L) holds.

2. Let $\xi = \phi_0(0)$, and let $\phi \in X_{f,a}$ be given. Set $x = x^\phi$. For $0 \le t \le a$, we have

$$|x(t) - \xi| = |x(0) - \xi + \int_0^t x'(s)ds| = |x(0) - \xi + \int_0^t f(x_s)ds|$$

$$\le \|x_0 - \phi_0\|_C + L\int_0^t \|x_s - \phi_0\|_C ds,$$

as $f(\phi_0) = 0$. For any $t \in [0,a]$ there exists $t_0 \in [t-h,t]$ with $\|x_t - \phi_0\|_C = |x(t_0) - \xi|$. In case $t_0 < 0$ we obtain

$$\|x_t - \phi_0\|_C \le \|x_0 - \phi_0\|_C$$

while in case $t_0 \ge 0$,

$$\|x_t - \phi_0\|_C \le \|x_0 - \phi_0\|_C + L\int_0^{t_0} \|x_s - \phi_0\|_C ds$$

$$\le \|x_0 - \phi_0\|_C + L\int_0^t \|x_s - \phi_0\|_C ds.$$

In both cases,

$$\|x_t - \phi_0\|_C \le \|x_0 - \phi_0\|_C + L\int_0^t \|x_s - \phi_0\|_C ds.$$

Gronwall's lemma yields

$$\|F(t,\phi) - \phi_0\|_C \le \|\phi - \phi_0\|_C \, e^{Lt} \text{ on } [0,a] \times X_{f,a}.$$

Consequently, for all $(t,\phi) \in [0,a] \times X_{f,a}$,

$$|(x^\phi)'(t)| = |f(x_t^\phi) - f(\phi_0)| \le L\|x_t^\phi - \phi_0\|_C \le L\, e^{Lt}\|\phi - \phi_0\|_C.$$

It follows that

$$\|(x_t^\phi)' - (\phi_0)'\|_C \le L\, e^{Lt}\|\phi - \phi_0\|_{C^1},$$

and finally

$$\|F(t,\phi) - \phi_0\|_{C^1} \le (L+1)e^{La}\|\phi - \phi_0\|_{C^1}$$

on $[0,a] \times X_{f,a}$. □

Now we can prove the desired invariance property of W_s with respect to the semiflow F, and an exponential estimate. Set

$$\gamma = -\frac{\log\,\alpha}{a};$$

then $0 < \beta < \gamma$.

Proposition 3.5.4. *There exists an open neighbourhood W^s of ϕ_0 in W_s so that $[0,\infty) \times W^s \subset \Omega$, $F([0,\infty) \times W^s) \subset W_s$, and there is a constant $c_w \ge 0$ so that*

$$\|F(t,\psi) - \phi_0\|_{C^1} \le c_w \, e^{-\gamma\, t}\|\psi - \phi_0\|_{C^1} \textit{ for all } \psi \in W^s,\ t \ge 0. \tag{3.5.1}$$

Proof. 1. Set $V_N = R(N)$. Choose c_s according to Corollary 3.5.2 (ii) and $X_{f,a}$ and c_a according to Proposition 3.5.3. It follows that there is an open neighbourhood W^s of ϕ_0 in $W_s \cap X_{f,a} \subset V_N \cap X_{f,a}$ so that

$$F_a(W^s) \subset W^s, \tag{3.5.2}$$

$$F([0,a] \times W^s) \subset V_N, \tag{3.5.3}$$

and

$$\{\chi \in Y : \|\chi\|_{C^1} \le \|P\|_{C^1}\, c_a\, c_s \sup_{\eta \in W^s} \|\eta - \phi_0\|_{C^1}\} \subset N. \tag{3.5.4}$$

Using (3.5.2) and (3.5.3) and properties of the semiflow we get $[0,\infty) \times W^s \subset \Omega$ and $F([0,\infty) \times W^s) \subset V_N$.

2. Proof of $F([0,\infty) \times W^s) \subset W_s$. Let $t \ge 0$ and $\psi \in W^s$ be given. The assertion $\rho = F(t,\psi) \in W_s$ is equivalent to

$$K(\rho) \in K(W_s) = W. \tag{3.5.5}$$

By the remarks in part 1 the point ρ defines a trajectory $(\rho_j)_0^\infty$ of F_a in V_N, with $\rho_0 = \rho$, and the point $\psi \in W^s$ defines a trajectory $(\psi_j)_0^\infty$ of F_a in $W^s \subset V_N$, with $\psi_0 = \psi$. The points $\chi_j = K(\rho_j)$ form a trajectory of H in N since

$$\chi_{j+1} = K(\rho_{j+1}) = K(F_a(\rho_j)) = K(F_a(R(\chi_j))) = H(\chi_j)$$

for all integers $j \ge 0$. Proposition 3.5.1 (iv) shows that (3.5.5) follows from

$$\lambda^{-j}\chi_j \in N \text{ for all integers } j \ge 0. \tag{3.5.6}$$

Proof of (3.5.6). Let $j \in \mathbb{N}_0$, $k \in \mathbb{N}_0$, $ka \le t < (k+1)a$. Then

$$\begin{aligned}
\|\chi_j\|_{C^1} &= \|K(\rho_j)\|_{C^1} \\
&= \|P(\rho_j - \phi_0)\|_{C^1} \\
&\le \|P\|_{C^1}\|\rho_j - \phi_0\|_{C^1} \\
&= \|P\|_{C^1}\|F(ja,\rho) - \phi_0\|_{C^1} \\
&= \|P\|_{C^1}\|F(t-ka, F((j+k)a,\psi)) - \phi_0\|_{C^1} \\
&\le \|P\|_{C^1}\, c_a\, \|\psi_{j+k} - \phi_0\|_{C^1} \\
&\le \|P\|_{C^1}\, c_a\, c_s\, \alpha^{j+k}\|\psi - \phi_0\|_{C^1},
\end{aligned}$$

hence

$$\|\lambda^{-j}\chi_j\|_{C^1} \le \|P\|_{C^1}\, c_a\, c_s \cdot 1 \cdot \sup_{\eta \in W^s} \|\eta - \phi_0\|_{C^1},$$

which yields

$$\lambda^{-j}\chi_j \in N,$$

according to (3.5.4).

3. Proof of (3.5.1). Let $\psi \in W^s$, $t \ge 0$, $j \in \mathbb{N}_0$, $ja \le t < (j+1)a$. Then

$$\begin{aligned}
\|F(t,\psi) - \phi_0\|_{C^1} &= \|F(t-ja, F(ja,\psi)) - \phi_0\|_{C^1} \\
&\le c_a\|F(ja,\psi) - \phi_0\|_{C^1} \\
&\le c_a\, c_s\, \alpha^j\, \|\psi - \phi_0\|_{C^1} \\
&= c_a\, c_s\, e^{j\,\log\alpha}\|\psi - \phi_0\|_{C^1} \\
&= c_a\, c_s\, e^{(t\,\log\alpha)/a} e^{(j-\frac{t}{a})\log\alpha}\|\psi - \phi_0\|_{C^1} \\
&\le c_a\, c_s\, e^{(t\,\log\alpha)/a}\alpha^{-1}\|\psi - \phi_0\|_{C^1}.
\end{aligned}$$

□

The C^1-submanifold W^s of X_f is the local stable manifold of F at ϕ_0. It is locally positively invariant under F, with tangent space

$$T_{\phi_0}W^s = C_s \cap T_{\phi_0}X_f,$$

and it has the following uniqueness property: There exists a constant $c > 0$ so that all initial data $\psi \in X_f$ with $[0,\infty) \times \{\psi\} \subset \Omega$ and

$$e^{\beta t}\|F(t,\psi) - \phi_0\|_{C^1} < c \text{ for all } t \ge 0$$

belong to W^s. This property is easily established by means of estimates as in the preceding proofs.

We said already that analogously to the approach to local *stable* manifolds just presented one obtains continuously differentiable local *unstable* manifolds for the semiflow from local unstable manifolds of the maps F_a, $a > 0$.

Local *unstable* manifolds for certain classes of differential equations with state-dependent delay were also obtained in earlier work, by Krishnan [128, 129] and more generally in [131]. The proofs in [128, 129] and [131] proceed without knowledge of a semiflow and use the heuristic approach with the auxiliary linear equation mentioned in subsection 3.4. It is remarkable that in [131] higher order differentiability of local unstable manifolds is achieved.

Related to work on local invariant manifolds is a result of Arino and Sanchez [10] about saddle point behaviour of solutions close to a stationary point which is hyperbolic, i.e., there is no spectrum of the generator G_e on the imaginary axis. Also in [10] no semiflow is available, and the auxiliary linear equation is used.

3.6. **The principle of linearized stability.** For $f : U \to \mathbb{R}^n$, $U \subset C^1$ open, with property (S) and the associated semiflow F of Theorem 3.2.1 the results of subsection 3.4 and Proposition 3.5.4 yield the following Principle of Linearized Stability.

Theorem 3.6.1. *If all eigenvalues of G_e have negative real part then ϕ_0 is exponentially asymptotically stable for the semiflow F.*

For applications, recall that the eigenvalues of G_e and their multiplicities are given by the familiar transcendental characteristic equation which is obtained from the Ansatz $v(t) = e^{\lambda\, t}c$ for a solution to the equation

$$v'(t) = D_e f(\phi_0) v_t.$$

Earlier Principles of Linearized Stability, for certain classes of differential equations with state-dependent delay, are due to Cooke and Huang [49] and to Hartung and Turi [107, 110]. The proofs employ the heuristic approach described in Subsection 3.4. Related is work by Hartung [103] on exponential stability of periodic solutions to nonautonomous, periodic differential equations with state-dependent delay, and a result by Győri and Hartung [94] who derive exponential stability of the zero solution of the nonautonomous equation

$$x'(t) = a(t)\, x(t - r(t, x(t)))$$

from exponential stability of the zero solution to the linear nonautonomous RFDE

$$y'(t) = a(t)\, y(t - r(t, 0)).$$

4. Center manifolds

4.1. **Preliminaries.** Assume $f : U \to \mathbb{R}^n$, $U \subset C^1$ open, with property (S) of Section 3, and define X_f as in Section 3. By Theorem 3.2.1, in case $X_f \neq \emptyset$ the solutions of the equation

$$x'(t) = f(x_t) \tag{1.0.1}$$

with initial function $x_0 = \phi \in X_f$ define a semiflow $F : \Omega \to X_f$. Assume that $0 \in U$ and 0 is a stationary point of F. Define $L = Df(0)$, $L_e = D_e f(0)$ and

$$r : U \ni \phi \mapsto f(\phi) - L\phi \in \mathbb{R}^n.$$

Clearly, r also satisfies (S), and $r(0) = 0$, $Dr(0) = 0$. Then (1.0.1) is equivalent to

$$x'(t) = Lx_t + r(x_t). \tag{4.1.1}$$

The solutions of the IVP

$$y'(t) = L_e y_t, \ y_0 = \phi \in C$$

define the strongly continuous semigroup $(T_e(t))_{t\geq 0}$ on C with generator G_e : $\mathrm{dom}(G_e) \to C$, $\mathrm{dom}(G_e) = \{\phi \in C^1 : \phi'(0) = L_e(\phi)\}$, $G_e\phi = \phi'$. The realified generalized eigenspaces of G_e given by the eigenvalues with negative, zero and positive real part are the stable C_s, center C_c and unstable C_u spaces, respectively. We have the decomposition

$$C = C_s \oplus C_c \oplus C_u,$$

C_s is infinite dimensional, C_c and C_u are finite dimensional, $C_c \subset \mathrm{dom}(G_e)$, $C_u \subset \mathrm{dom}(G_e)$. The set $C_s^1 = C_s \cap C^1$ is a closed subset of C^1. Hence we get the decomposition

$$C^1 = C_s^1 \oplus C_c \oplus C_u \tag{4.1.2}$$

of C^1.

The derivatives $D_2F(t, 0)$, $t \geq 0$, form the strongly continuous semigroup $T(t)$, $t \geq 0$, on $T_0X_f = \mathrm{dom}(G_e)$ with generator G. The stable, center and unstable subspaces of G are $C_s \cap \mathrm{dom}(G_e) = C_s^1 \cap \mathrm{dom}(G_e)$, C_c and C_u, respectively, and

$$T_0X_f = \mathrm{dom}(G_e) = (C_s^1 \cap \mathrm{dom}(G_e)) \oplus C_c \oplus C_u.$$

In the sequel we assume

$$\dim C_c \geq 1.$$

The main result of this section guarantees the existence of a Lipschitz smooth (local) center manifold of F at the stationary point 0.

Theorem 4.1.1. *There exist open neighbourhoods $C_{c,0}$ of 0 in C_c and $C^1_{su,0}$ in $C_s^1 \oplus C_u$ with $N = C_{c,0} + C^1_{su,0} \subset U$, a Lipschitz continuous map $w_c : C_{c,0} \to C^1_{su,0}$ such that $w_c(0) = 0$ and for the graph*

$$W_c = \{\phi + w_c(\phi) : \phi \in C_{c,0}\}$$

of w_c the following holds.

(i) *$W_c \subset X_f$, and W_c is a $\dim C_c$-dimensional Lipschitz smooth submanifold of X_f.*
(ii) *If $x : \mathbb{R} \to \mathbb{R}^n$ is a continuously differentiable solution of (1.0.1) on $\mathbb{R}$ with $x_t \in N$ for all $t \in \mathbb{R}$, then $x_t \in W_c$ for all $t \in \mathbb{R}$.*

(iii) *W_c is locally positively invariant with respect to the semiflow F, i.e., if $\phi \in W_c$ and $\alpha > 0$ such that $F(t,\phi)$ is defined for all $t \in [0,\alpha)$, and $F(t,\phi) \in N$ for all $t \in [0,\alpha)$, then $F(t,\phi) \in W_c$ for all $t \in [0,\alpha)$.*

The proof will be given in Subsection 4.3. We follow the approach of [55] by applying the Lyapunov–Perron method. The variation-of-constants formula is from [55] which requires dual spaces and adjoint operators. Other forms of the variation-of-constants formula, obtained e.g. via integrated semigroups or extrapolation theory, could also be used. The existence of a global center manifold for a modified version of Equation (4.1.1) is formulated as a fixed point problem with a parameter. However, as the right hand side of (4.1.1) has smoothness properties only in the space C^1, the space, where we look for fixed points, should contain smoother functions than the corresponding space in [55]. This is why the proof is not a straightforward application of the technique of [55]. The C^1-smoothness of W_c also holds under the hypotheses of Theorem 4.1.1. A proof can be found in [133].

4.2. **The linear inhomogeneous equation.** Let $|\cdot|$ be a norm in $\mathbb{R}^n$. The spaces C, C^1 and their norms are defined as in Section 1. Let $L^\infty(-h,0;\mathbb{R}^n)$ denote the Banach space of measurable and essentially bounded functions from $(-h,0)$ into $\mathbb{R}^n$ equipped with the essential least upper bound norm $||\cdot||_\infty$.

We denote dual spaces and adjoint operators by an asterisk $*$ in the sequel. The elements $\phi^\odot$ of C^* for which the curve

$$[0,\infty) \ni t \mapsto T_e^*(t)\phi^\odot \in C^*$$

is continuous form a closed subspace $C^\odot$ (of C^*) which is positively invariant under $T_e^*(t)$, $t \geq 0$. The operators

$$T_e^\odot(t) : C^\odot \ni \phi^\odot \mapsto T_e^*(t)\phi^\odot \in C^\odot,\ t \geq 0,$$

constitute a strongly continuous semigroup on $C^\odot$. Similarly, we can introduce the dual space $C^{\odot*}$ and the semigroup of adjoint operators $T_e^{\odot*}(t)$, $t \geq 0$, which is strongly continuous on $C^{\odot\odot}$. There is an isometric isomorphism between $\mathbb{R}^n \times L^\infty(-h,0;\mathbb{R}^n)$ equipped with the norm $||(\alpha,\phi)|| = \max\{|\alpha|, ||\phi||_\infty\}$ and $C^{\odot*}$. We will identify $C^{\odot*}$ with $\mathbb{R}^n \times L^\infty(-h,0;\mathbb{R}^n)$ and omit the isomorphism. The original state space is sun-reflexive in the sense that, for the norm-preserving linear map $j : C \to C^{\odot*}$ given by $j(\phi) = (\phi(0),\phi)$, we have $j(C) = C^{\odot\odot}$. We also omit the embedding operator j and identify C and $C^{\odot\odot}$. All of these results as well as the decomposition of $C^{\odot*}$ and the variation-of-constants formula can be found in [55].

Let $Y^{\odot*}$ denote the subspace $\mathbb{R}^n \times \{0\}$ of $C^{\odot*}$. For the k-th unit vector e_k in $\mathbb{R}^n$ set $r_k^{\odot*} = (e_k, 0) \in Y^{\odot*}$. Let $l : \mathbb{R}^n \to Y^{\odot*}$ be the linear map given by $l(e_k) = r_k^{\odot*}$, $k \in \{1,2,\ldots,n\}$. Then l has an inverse l^{-1}, and $||l|| = ||l^{-1}|| = 1$.

Let $G_e^{\odot*}$ denote the generator of $T_e^{\odot*}$. For the spectra $\sigma(G_e)$ and $\sigma(G_e^{\odot*})$ we have $\sigma(G_e) = \sigma(G_e^{\odot*})$. Recall that we assumed

$$\sigma(G_e) \cap i\mathbb{R} \neq \emptyset.$$

Then $C^{\odot*}$ can be decomposed as

$$C^{\odot*} = C_s^{\odot*} \oplus C_c \oplus C_u, \tag{4.2.1}$$

where $C_s^{\odot*}$, C_c, C_u are closed subspaces of $C^{\odot*}$, C_c and C_u are contained in C^1, $1 \leq \dim C_c < \infty$, $\dim C_u < \infty$. The subspaces $C_s^{\odot*}$, C_c and C_u are invariant under $T_e^{\odot*}(t)$, $t \geq 0$, and $T_e(t)$ can be extended to a one-parameter group on both C_c and

C_u. There exist real numbers $K \geq 1$, $a < 0$, $b > 0$ and $\epsilon > 0$ with $\epsilon < \min\{-a, b\}$ such that

$$\begin{aligned} ||T_e(t)\phi|| &\leq Ke^{bt}||\phi||, \qquad t \leq 0,\ \phi \in C_u, \\ ||T_e(t)\phi|| &\leq Ke^{\epsilon|t|}||\phi||, \qquad t \in \mathbb{R},\ \phi \in C_c, \\ ||T_e^{\odot *}(t)\phi|| &\leq Ke^{at}||\phi||, \qquad t \geq 0,\ \phi \in C_s^{\odot *}. \end{aligned} \tag{4.2.2}$$

Using the identification of C and $C^{\odot\odot}$, we obtain $C_s^1 = C^1 \cap C_s^{\odot *}$. The decompositions (4.1.2) and (4.2.1) define the projection operators P_s, P_c, P_u and $P_s^{\odot *}, P_c^{\odot *}, P_u^{\odot *}$ with ranges C_s^1, C_c, C_u and $C_s^{\odot *}, C_c, C_u$, respectively.

We need a variation-of-constants formula for solutions of

$$x'(t) = L_e x_t + q(t) \tag{4.2.3}$$

with a continuous function $q : \mathbb{R} \to \mathbb{R}^n$.

If c, d are reals with $c \leq d$, and $w : [c, d] \to C^{\odot *}$ is continuous, then the weak-star integral

$$\int_c^d T_e^{\odot *}(d - \tau)w(\tau)\, d\tau \in C^{\odot *}$$

is defined by

$$\left(\int_c^d T_e^{\odot *}(d - \tau)w(\tau)\, d\tau\right)(\phi^{\odot}) = \int_c^d T_e^{\odot *}(d - \tau)w(\tau)(\phi^{\odot})\, d\tau$$

for all $\phi^{\odot} \in C^{\odot}$.

If $I \subset \mathbb{R}$ is an interval, $q : I \to \mathbb{R}^n$ is continuous and $x : I + [-h, 0] \to \mathbb{R}^n$ is a solution of (4.2.3) on I, then the curve $u : I \ni t \mapsto x_t \in C$ satisfies the integral equation

$$u(t) = T_e(t - s)u(s) + \int_s^t T_e^{\odot *}(t - \tau)Q(\tau)\, d\tau, \qquad t, s \in I,\ s \leq t, \tag{4.2.4}$$

with $Q(t) = l(q(t))$, $t \in I$. Moreover, if $Q : I \to Y^{\odot *}$ is continuous, and $u : I \to C$ satisfies (4.2.4), then there is a continuous function $x : I + [-h, 0] \to \mathbb{R}^n$ such that $x_t = u(t)$ for all $t \in I$, and x satisfies (4.2.3) with $q(t) = l^{-1}(Q(t))$, $t \in I$. So, there is a one-to-one correspondence between the solutions of (4.2.3) and (4.2.4).

For a Banach space B with norm $||\cdot||$ and a real $\eta \geq 0$, we define the Banach space

$$C_\eta(\mathbb{R}, B) = \left\{ b \in C(\mathbb{R}, B) : \sup_{t \in \mathbb{R}} e^{-\eta|t|}||b(t)|| < \infty \right\}$$

with norm

$$||b||_{C_\eta^0(\mathbb{R},B)} = \sup_{t \in \mathbb{R}} e^{-\eta|t|}||b(t)||.$$

For $\eta \geq 0$, we introduce the notation

$$Y_\eta = C_\eta(\mathbb{R}, Y^{\odot *}),\ C_\eta^0 = C_\eta(\mathbb{R}, C),\ C_\eta^1 = C_\eta(\mathbb{R}, C^1).$$

We need the following smoothing property of Equation (4.2.4).

Proposition 4.2.1. *Let $\eta \geq 0$, $Q \in Y_\eta$, $u \in C_\eta^0$, and assume that u satisfies*

$$u(t) = T_e(t - s)u(s) + \int_s^t T_e^{\odot *}(t - \tau)Q(\tau)\, d\tau, \qquad -\infty < s \leq t < \infty.$$

Then $u \in C^1_\eta$ *and*

$$||u||_{C^1_\eta} \le (1 + e^{\eta h}||L_e||)||u||_{C^0_\eta} + e^{\eta h}||Q||_{Y_\eta}.$$

Proof. Define $q : \mathbb{R} \to \mathbb{R}^n$ by $q(t) = l^{-1}(Q(t))$, $t \in \mathbb{R}$. Then $q \in C_\eta(\mathbb{R}, \mathbb{R}^n)$, and

$$||q||_{C_\eta(\mathbb{R},\mathbb{R}^n)} = ||Q||_{Y_\eta}.$$

The function $x : \mathbb{R} \to \mathbb{R}^n$, given by $x(t) = u(t)(0)$, satisfies $x_t = u(t)$, $t \in \mathbb{R}$, and Equation (4.2.3) holds for all $t \in \mathbb{R}$. Then x is C^1-smooth, $x_t \in C^1$ for all $t \in \mathbb{R}$, and the mapping $\mathbb{R} \ni t \mapsto x_t \in C^1$ is continuous. Moreover, for all $t \in \mathbb{R}$,

$$\begin{aligned}
|x'(t)| &\le ||L_e||\,||x_t||_C + |q(t)| \\
&= ||L_e||\,||u(t)||_C + ||Q(t)||_{Y^{\odot *}} \\
&\le e^{\eta|t|}\left(||L_e||\,||u||_{C^0_\eta} + ||Q||_{Y_\eta}\right).
\end{aligned}$$

Hence

$$\begin{aligned}
\sup_{t\in\mathbb{R}} e^{-\eta|t|}||x'_t||_C &= \sup_{t\in\mathbb{R}} e^{-\eta|t|} \sup_{-h\le s\le 0} |x'(t+s)| \\
&\le \left(||L_e||\,||u||_{C^0_\eta} + ||Q||_{Y_\eta}\right) \sup_{t\in\mathbb{R}} e^{-\eta|t|} \sup_{-h\le s\le 0} e^{\eta|t+s|} \\
&\le e^{\eta h}\left(||L_e||\,||u||_{C^0_\eta} + ||Q||_{Y_\eta}\right).
\end{aligned}$$

Therefore, $u \in C^1_\eta$, and

$$\begin{aligned}
||u||_{C^1_\eta} &= \sup_{t\in\mathbb{R}} e^{-\eta|t|}||x_t||_{C^1} = \sup_{t\in\mathbb{R}} e^{-\eta|t|}\left(||x_t||_C + ||x'_t||_C\right) \\
&\le ||u||_{C^0_\eta} + e^{\eta h}||L_e||\,||u||_{C^0_\eta} + e^{\eta h}||Q||_{Y_\eta}.
\end{aligned}$$

□

For a given $Q : \mathbb{R} \to Y^{\odot *}$ we (formally) define

$$\begin{aligned}
(\mathcal{K}Q)(t) = &\int_0^t T_e^{\odot *}(t-\tau)P_c^{\odot *}Q(\tau)\,d\tau + \int_\infty^t T_e^{\odot *}(t-\tau)P_u^{\odot *}Q(\tau)\,d\tau \\
&+ \int_{-\infty}^t T_e^{\odot *}(t-\tau)P_s^{\odot *}Q(\tau)\,d\tau.
\end{aligned}$$

Proposition 4.2.2. *Assume* $\eta \in (\epsilon, \min\{-a, b\})$. *Then the mapping*

$$\mathcal{K}_\eta : Y_\eta \ni Q \mapsto \mathcal{K}Q \in C^1_\eta$$

is linear bounded with norm

$$||\mathcal{K}_\eta|| \le K\left(1 + e^{\eta h}||L_e||\right)\left(\frac{1}{\eta-\epsilon} + \frac{1}{-a-\eta} + \frac{1}{b-\eta}\right) + e^{\eta h}.$$

If $Q \in Y_\eta$ *then* $u = \mathcal{K}Q$ *is the unique solution of*

$$u(t) = T_e(t-s)u(s) + \int_s^t T_e^{\odot *}(t-\tau)Q(\tau)\,d\tau, \qquad -\infty < s \le t < \infty, \tag{4.2.5}$$

in C^1_η *with* $P_c^{\odot *}u(0) = 0$.

Proof. Lemma IX.3.2 of [55] shows that $\mathcal{K}$ as a mapping from Y_η into C^0_η is linear bounded such that its norm is bounded by

$$K\left(\frac{1}{\eta-\epsilon}+\frac{1}{-a-\eta}+\frac{1}{b-\eta}\right),$$

moreover $u=\mathcal{K}Q$ with $Q\in Y_\eta$ is the unique solution of (4.2.5) with $P_c^{\odot *}u(0)=0$. Hence Proposition 4.2.1 yields the boundedness of $\mathcal{K}_\eta$ with the stated bound for the norm. □

4.3. **Construction of a center manifold.** Now we prove Theorem 4.1.1.

As $\dim C_c<\infty$, there is a norm $|\cdot|_c$ on C_c which is C^∞-smooth on $C_c\setminus\{0\}$. Then

$$|\phi|_1=\max\{|P_c\phi|_c,||(\mathrm{id}_{C^1}-P_c)\phi||_{C^1}\},\qquad \phi\in C^1,$$

defines the new norm $|\cdot|_1$ on C^1 which is equivalent to $||\cdot||_{C^1}$.

Let $\rho:\mathbb{R}\to\mathbb{R}$ be a C^∞-smooth function so that $\rho(t)=1$ for $t\le 1$, $\rho(t)=0$ for $t\ge 2$, and $\rho(t)\in(0,1)$ for $t\in(1,2)$.

Define

$$\hat r(\phi)=\begin{cases} r(\phi), & \text{if } \phi\in U;\\ 0, & \text{if } \phi\notin U.\end{cases}$$

For any $\delta>0$, let

$$r_\delta(\phi)=\hat r(\phi)\rho\left(\frac{|P_c\phi|_c}{\delta}\right)\rho\left(\frac{|(\mathrm{id}_{C^1}-P_c)\phi|_1}{\delta}\right),\quad \phi\in C^1.$$

For $\gamma>0$ set $B_\gamma(C^1)=\{\phi\in C^1:|\phi|_1<\gamma\}$.

Choose $\delta_0>0$ so that

$$B_{2\delta_0}(C^1)\subset U,$$

and $r|_{B_{2\delta_0}(C^1)}$, $Dr|_{B_{2\delta_0}(C^1)}$ are bounded. Then, for any $\delta\in(0,\delta_0)$

$$r_\delta|_{\{\phi\in C^1:|(\mathrm{id}_{C^1}-P_c)\phi|_1<\delta\}}(\phi)=\hat r(\phi)\rho\left(\frac{|P_c\phi|_c}{\delta}\right),\quad \phi\in C^1,$$

and $r_\delta|_{\{\phi\in C^1:|(\mathrm{id}_{C^1}-P_c)\phi|_1<\delta\}}$ is a bounded and C^1-smooth function with bounded derivative.

There exist $\delta_1\in(0,\delta_0)$ and a nondecreasing function $\mu:[0,\delta_1]\to[0,1]$ such that μ is continuous at 0, $\mu(0)=0$, and for all $\delta\in(0,\delta_1]$ and for all $\phi,\psi\in C^1$

$$\begin{aligned} |r_\delta(\phi)|&\le\delta\mu(\delta),\\ |r_\delta(\phi)-r_\delta(\psi)|&\le\mu(\delta)||\phi-\psi||_{C^1}.\end{aligned}\tag{4.3.1}$$

For a proof of completely analogous estimates see e.g. Proposition II.2 in [135].

For $\delta\in(0,\delta_1]$ we consider the modified equations

$$x'(t)=Lx_t+r_\delta(x_t),\qquad t\in\mathbb{R},\tag{4.3.2}$$

and

$$u(t)=T_e(t-s)+\int_s^t T_e^{\odot *}(t-\tau)l(r_\delta(u(\tau)))\,d\tau,\qquad -\infty<s\le t<\infty.\tag{4.3.3}$$

These equations are equivalent in the following sense: If $x:\mathbb{R}\to\mathbb{R}^n$ is C^1-smooth and is a solution of Equation (4.3.2), then $u:\mathbb{R}\ni t\mapsto x_t\in C^1$ is a solution of Equation (4.3.3), and conversely, a continuous $u:\mathbb{R}\to C^1$ satisfying (4.3.3) defines a C^1-smooth solution of (4.3.2) by $x(t)=u(t)(0)$, $t\in\mathbb{R}$.

Now we fix the reals $\eta \in (\epsilon, \min\{-a, b\})$ and $\delta \in (0, \delta_1)$ such that

$$||\mathcal{K}_\eta||\mu(\delta) < \frac{1}{2}. \tag{4.3.4}$$

Let the substitution operator

$$R : (C^1)^{\mathbb{R}} \to (Y^{\odot *})^{\mathbb{R}}$$

of the map $C^1 \ni \phi \mapsto l(r_\delta(\phi)) \in Y^{\odot *}$ be given by

$$R(u)(t) = l(r_\delta(u(t))).$$

(4.3.1) and $||l|| = 1$ yield that

$$R(C^1_\eta) \subset Y_\eta,$$

and for the induced map

$$R_{\delta\eta} : C^1_\eta \to Y_\eta,$$

the inequalities

$$||R_{\delta\eta}(u)||_{Y_\eta} \le \delta\mu(\delta), \qquad u \in C^1_\eta,$$

and

$$||R_{\delta\eta}(u) - R_{\delta\eta}(v)||_{Y_\eta} \le \mu(\delta)||u - v||_{C^1_\eta}, \qquad u, v \in C^1_\eta, \tag{4.3.5}$$

hold.

Let the mapping $S : C_c \to C^1_\eta$ be given by $(S\phi)(t) = T_e(t)\phi$, $\phi \in C_c$, $t \in \mathbb{R}$. For all $\phi \in C_c$ we have $||T_e(t)\phi||_{C^1} = ||T_e(t)\phi||_C + ||\frac{d}{dt}(T_e(t)\phi)||_C$ and $\frac{d}{dt}(T_e(t)\phi) = T_e(t)G_e\phi = T_e(t)\phi'$. Therefore, by applying the second inequality in (4.2.2) and $\eta > \epsilon$, we find

$$\begin{aligned} ||S\phi||_{C^1_\eta} &= \sup_{t\in\mathbb{R}} e^{-\eta|t|} \left(||T_e(t)\phi||_C + ||T_e(t)\phi'||_C\right) \\ &\le K \left(||\phi||_C + ||\phi'||_C\right) \\ &\le K||\phi||_{C^1}. \end{aligned} \tag{4.3.6}$$

Define the mapping

$$\mathcal{G} : C^1_\eta \times C_c \to C^1_\eta$$

by

$$\mathcal{G}(u, \phi) = S\phi + \mathcal{K}_\eta \circ R_{\delta\eta}(u), \qquad u \in C^1_\eta,\ \phi \in C_c.$$

For all $u, v \in C^1_\eta$ and $\phi \in C_c$, (4.3.4) and (4.3.5) yield

$$\begin{aligned} ||\mathcal{G}(u, \phi) - \mathcal{G}(v, \phi)||_{C^1_\eta} &\le ||\mathcal{K}_\eta||\, ||R_{\delta\eta}(u) - R_{\delta\eta}(v)||_{Y_\eta} \\ &\le ||\mathcal{K}_\eta||\mu(\delta)||u - v||_{C^1_\eta} \\ &\le \frac{1}{2}||u - v||_{C^1_\eta}. \end{aligned}$$

If $\gamma > 0$ and $\phi \in C_c$ with $||\phi||_{C^1} \le \gamma/(2K)$, and $u \in \overline{B_\gamma(C^1_\eta)}$, then, by using (4.3.4), (4.3.5) and (4.3.6),

$$\begin{aligned} ||\mathcal{G}(u, \phi)||_{C^1_\eta} &\le K||\phi||_{C^1} + ||\mathcal{K}_\eta||\mu(\delta)||u||_{C^1_\eta} \\ &\le \frac{\gamma}{2} + \frac{\gamma}{2} = \gamma. \end{aligned}$$

Therefore, $\mathcal{G}(\cdot, \phi)$ maps $\overline{B_\gamma(C^1_\eta)}$ into itself provided $\gamma \ge 2K||\phi||_{C^1}$. In addition, $\mathcal{G}(\cdot, \phi)$ is Lipschitz continuous with Lipschitz constant 1/2.

Consequently, there is a map

$$u^* : C_c \to C^1_\eta$$

such that, for each $\phi \in C_c$, $u = u^*(\phi)$ is the unique solution in C^1_η of the equation

$$u = \mathcal{G}(u, \phi).$$

The mapping u^* is globally Lipschitz continuous since

$$||u^*(\phi) - u^*(\psi)||_{C^1_\eta} \le K||\phi - \psi||_{C^1} + \frac{1}{2}||u^*(\phi) - u^*(\psi)||_{C^1_\eta},$$

yielding

$$||u^*(\phi) - u^*(\psi)||_{C^1_\eta} \le 2K||\phi - \psi||_{C^1}$$

for all $\phi, \psi \in C_c$.

The set

$$W = \{u^*(\phi)(0) : \phi \in C_c\}$$

is called the global center manifold of Equation (4.3.2) at the stationary point 0.

Setting

$$w : C_c \ni \phi \mapsto (\mathrm{id}_{C^1} - P_c)u^*(\phi)(0) \in C^1_s \oplus C_u,$$

we get the graph representation

$$W = \{\phi + w(\phi) : \phi \in C_c\}$$

for W.

For all $\phi \in C_c$ we have

$$\begin{aligned}
|w(\phi)|_1 &= ||w(\phi)||_{C^1} = ||(\mathrm{id}_{C^1} - P_c)u^*(\phi)(0)||_{C^1} \\
&= ||\mathcal{K}_\eta(R_{\delta\eta}(u^*(\phi)))(0)||_{C^1} \le ||\mathcal{K}_\eta(R_{\delta\eta}(u^*(\phi)))||_{C^1_\eta} \\
&\le ||\mathcal{K}_\eta||\,||R_{\delta\eta}(u^*(\phi))||_{Y_\eta} \\
&\le ||\mathcal{K}_\eta||\delta\mu(\delta) < \delta.
\end{aligned}$$

An important consequence is that

$$W \subset \{\phi \in C^1 : |(\mathrm{id}_{C^1} - P_c)\phi|_1 < \delta\},$$

that is, W is contained in the δ-neighbourhood of C_c, where r_δ is C^1-smooth with bounded derivative. This fact is essential in the proof of the C^1-smoothness of the center manifold.

Setting

$$\begin{aligned}
C_{c,0} &= \{\phi \in C_c : |\phi|_1 < \delta\}, \\
C^1_{su,0} &= \{\phi \in C^1_s \oplus C_u : |\phi|_1 < \delta\}, \\
N = C_{c,0} + C^1_{su,0} &= \{\phi \in C^1 : |\phi|_1 < \delta\}, \\
w_c &= w|_{C_{c,0}}, \\
W_c &= \{\phi + w_c(\phi) : \phi \in C_{c,0}\},
\end{aligned}$$

we obtain that $w_c(C_{c,0}) \subset C^1_{su,0}$ and w_c is Lipschitz continuous. As $\mathcal{G}(0,0) = 0$, it follows that $u^*(0) = 0$, and consequently $w_c(0) = 0$.

Let $v \in C^1_\eta$ be a solution of Equation (4.3.3). Define $z : \mathbb{R} \to C^1$ by

$$z(t) = v(t) - T_e(t)P_cv(0), \qquad t \in \mathbb{R}.$$

Obviously, $v \in C^1_\eta$ implies $z \in C^1_\eta$. Moreover,

$$z(t) = T_e(t-s)z(s) + \int_s^t T_e^{\odot *}(t-\tau)l(r_\delta(v(\tau)))\,d\tau, \qquad -\infty < s \le t < \infty.$$

As $P_c z(0) = 0$ and $v \in C^1_\eta$, Proposition 4.2.2 yields

$$z = \mathcal{K}_\eta(R_{\delta\eta}(v)).$$

Therefore

$$v = S(P_c v(0)) + \mathcal{K}_\eta(R_{\delta\eta}(v)),$$

and

$$v(0) = u^*(P_c v(0)) \in W^c.$$

For any $t \in \mathbb{R}$ and $\hat{v} : \mathbb{R} \ni s \mapsto v(t+s) \in C^1$, it is clear that $\hat{v} \in C^1_\eta$, and $\hat{v}$ is also a solution of Equation (4.3.3). Therefore, $v(t) = \hat{v}(0) \in W$ follows for all $t \in \mathbb{R}$. Consequently, for each $v \in C^1_\eta$ satisfying Equation (4.3.3), $v(t) \in W$ holds for all $t \in \mathbb{R}$.

If $x : \mathbb{R} \to \mathbb{R}^n$ is a solution of Equation (1.0.1) with $x_t \in N$ for all $t \in \mathbb{R}$, then (4.3.2) also holds, and $u(t) = x_t$, $t \in \mathbb{R}$, satisfies Equation (4.3.3) since $r|_N = r_\delta|_N$, and $u \in C^1_\eta$. Thus, $x_t \in W$, $t \in \mathbb{R}$. This proves (ii) of Theorem 4.1.1.

In order to show (iii) in Theorem 4.1.1, let $\phi \in W_c$. Then $u^*(P_c\phi) \in C^1_\eta$, and $u^*(P_c\phi)(t) \in W$ for all $t \in \mathbb{R}$. Let $\beta \in (0,\infty]$ be maximal so that

$$u^*(P_c\phi)(t) \in N \qquad \text{for all } t \in [0,\beta),$$

that is,

$$u^*(P_c\phi)(t) \in W_c \qquad \text{for all } t \in [0,\beta).$$

Then there exists a C^1-smooth function $y : [-h,\beta) \to \mathbb{R}^n$ so that $y_t = u^*(P_c\phi)(t)$, $t \in [0,\beta)$, and

$$\begin{cases} y'(t) = Ly_t + r(y_t), & 0 < t < \beta, \\ y_0 = \phi. \end{cases}$$

If $x^\phi : [-h,\alpha)$ is also a solution of the above IVP with $x^\phi_t \in N$, $t \in [0,\alpha)$, then the result on unique continuation of solutions in Section 3 yields $\alpha \le \beta$ and $x^\phi(t) = y(t)$, $t \in [-h,\alpha)$.

For any $\phi \in W_c$, the function $x : \mathbb{R} \to \mathbb{R}^n$ defined by $x_t = u^*(P_c\phi)(t)$, $t \in \mathbb{R}$, is continuously differentiable and satisfies Equation (4.3.2). Consequently, $x'(t) = Lx_t + r_\delta(x_t)$, $t \in \mathbb{R}$. In particular, $\phi'(0) = L\phi + r_\delta(\phi)$. Using $\phi \in W_c \subset N$ and $r_\delta|_N = r|_N$, $\phi \in X_f$ follows. Thus, $W_c \subset X_f$.

Recall from Subsection 3.5 that there is an n-dimensional subspace $E \subset C^1$ which is a complement of $Y = T_0 X_f$ in C^1. If $e_1, \ldots, e_n$ is a basis for E, then using the decomposition $C^1 = C^1_s \oplus C_c \oplus C_u$ of C^1, for each $i \in \{1, \ldots, n\}$ we have

$$e_i = s_i + c_i + u_i$$

for some $s_i \in C^1_s$, $c_i \in C_c$, $u_i \in C_u$. As $C_c \oplus C_u \subset Y$, we have $s_i \notin Y$. Then the subspace $\hat{E}$ spanned by the vectors

$$\hat{e}_i = e_i - c_i - u_i, \qquad i \in \{1, \ldots, n\},$$

is also an n-dimensional complementary subspace of Y in C^1, and in addition $\hat{E} \subset C^1_s$. Therefore, without loss of generality, we may assume $E \subset C^1_s$. Then

$$C^1_s = E \oplus (C^1_s \cap Y),$$

$$Y = (C_s^1 \cap Y) \oplus C_c \oplus C_u,$$

and

$$C^1 = E \oplus (C_s^1 \cap Y) \oplus C_c \oplus C_u = E \oplus Y.$$

Let $P : C^1 \to C^1$ denote the projection along E onto Y. There is an open neighbourhood V of 0 in X_f so that $P : V \to Y$ is a manifold chart of X_f. Set $Y_0 = P(V)$. The inverse of $P : V \to Y_0$ is C^1-smooth. If $\delta > 0$ is sufficiently small, then $W_c \subset V$ and $PW_c \subset Y_0$. In order to complete the proof of (i) in Theorem 4.1.1, it is enough to show that PW_c is a $\dim C_c$-dimensional Lipschitz submanifold of Y. Indeed,

$$PW_c = \{P(\phi + w_c(\phi)) : \phi \in C_{c,0}\} = \{\phi + Pw_c(\phi) : \phi \in C_{c,0}\}.$$

As $w_c(\phi) \in C_s^1 \oplus C_u$, we have

$$Pw_c(\phi) \in (C_s^1 \cap Y) \oplus C_u.$$

Thus, PW_c is the graph of the Lipschitz continuous map

$$\{\phi \in C_c : |\phi|_1 < \delta\} \ni \chi \mapsto Pw_c(\chi) \in (C_s^1 \cap Y) \oplus C_u.$$

This completes the proof of Theorem 4.1.1.

4.4. **Discussion.** It is also true that the local center manifold W_c given in Theorem 4.1.1 is a C^1-submanifold of X_f. The proof will appear in [133]. It is based on the fact that for the global center manifold W of the modified Equation (4.3.1)

$$W \subset \{\phi \in C^1 : |(\mathrm{id}_{C^1} - P_c)\phi|_1 < \delta\},$$

and r_δ is C^1-smooth on the subset $\{\phi \in C^1 : |(\mathrm{id}_{C^1} - P_c)\phi|_1 < \delta\}$ of C^1 with bounded derivative. The techniques of [55] or [135] can be modified to our situation in order to show that the map $w : C_c \to C_s^1 \oplus C_u$ is C^1-smooth.

Another way to obtain C^1-smooth center manifolds is to consider, for some $a > 0$, the time-a map

$$F_a : \Omega_a \to X_f,$$

and construct a local center manifold of F_a at its fixed point 0. However, as a 2-dimensional ordinary differential equation example shows in [132], the obtained center manifold of F_a is not necessarily a locally invariant center manifold of the semiflow F. There is a standard technique to overcome this difficulty (see, e.g., [132]). The idea is that the modification of the map F_a should be done through the modification of the semiflow F. This requires a modification and extension of F from a small neighbourhood of $[0, a] \times \{0\}$ in Ω to a certain global semiflow. This is a nontrivial task. It is an open problem to work out the complete proof by using this approach.

Local bifurcation results for functional differential equations with state-dependent delay through the center manifold reduction would require C^k-smooth local center manifolds also with $k > 1$. As far as we know such results are not available at the moment.

A first step towards the proof of a C^k-smooth center manifold could be a C^k-smooth version of the results of Section 3. Then a C^k-smooth time-a map could be the basis to construct a C^k-smooth center manifold as suggested above for $k = 1$.

Another possible way is the extension of the approach explained in this section. Notice that it does not require the existence of a smooth semiflow. We remark that this idea worked for a construction of C^k-smooth unstable manifolds under natural conditions on f which are satisfied by equations with state-dependent delay [131].

5. Hopf bifurcation

Hopf bifurcation is the phenomenon that under certain conditions small periodic orbits appear close to a stationary point when in the underlying differential equation a parameter is varied and passes a critical value. In this section we state a Hopf bifurcation theorem for differential equations with state-dependent delay which has recently been proved by M. Eichmann [66]. The equation considered is a parametrized version of Equation (1.0.1), namely

$$x'(t) = g(\alpha, x_t). \tag{5.0.1}$$

The map $g : J \times U \to \mathbb{R}^n$ in Equation (5.0.1) is defined on the product of an open interval $J \subset \mathbb{R}$ and an open subset $U \subset C^1 = C^1([-h, 0], \mathbb{R}^n)$, with $h > 0$ and $n \in \mathbb{N}$. Let $C = C([-h, 0], \mathbb{R}^n)$ and let C^2 denote the Banach space of twice continuously differentiable functions $\phi : [-h, 0] \to \mathbb{R}^n$, with the norm given by $\|\phi\|_{C^2} = \|\phi\|_C + \|\phi'\|_C + \|\phi''\|_C$. The set

$$U^* = U \cap C^2$$

is an open subset of C^2. The following hypotheses on smoothness are assumed.

- **(H1):** The mapping $g : J \times U \to \mathbb{R}^n$ is continuously differentiable.
- **(H2):** For each $(\alpha, \phi) \in J \times U$ the partial derivative $D_2 g(\alpha, \phi)$ of g with respect to ϕ extends to a continuous linear map
$$D_{2,e} g(\alpha, \phi) : C \to \mathbb{R}^n.$$
- **(H3):** The mapping
$$J \times U \times C \ni (\alpha, \phi, \chi) \mapsto D_{2,e} g(\alpha, \phi)\chi \in \mathbb{R}^n$$
is continuous.
- **(H4):** The restriction $g^* = g|J \times U^*$ is twice continuously differentiable.
- **(H5):** For each $(\alpha, \phi) \in J \times U^*$ the second order partial derivative $D_2^2 g^*(\alpha, \phi) : C^2 \times C^2 \to \mathbb{R}^n$ of g^* with respect to ϕ has a continuous bilinear extension
$$D_{2,e}^2 g^*(\alpha, \phi) : C^1 \times C^1 \to \mathbb{R}^n.$$
- **(H6):** The mappings
$$J \times U^* \times C^1 \times C^1 \ni (\alpha, \phi, \chi_1, \chi_2) \mapsto D_{2,e}^2 g^*(\alpha, \phi)(\chi_1, \chi_2) \in \mathbb{R}^n$$
and
$$J \times U^* \times C^1 \ni (\alpha, \phi, \chi) \mapsto D_{2,e}^2 g^*(\alpha, \phi)(\chi, \cdot) \in L(C^2, \mathbb{R}^n)$$
are continuous.

The hypotheses (H1-H3) imply that each map $g(\alpha, \cdot) : U \to \mathbb{R}^n$, $\alpha \in J$, satisfies the hypotheses (S1-S3) of Theorem 3.2.1. Notice that condition (H3) is weaker than continuity of the map

$$J \times U \ni (\alpha, \phi) \mapsto D_{2,e} g(\alpha, \phi) \in L(C, \mathbb{R}^n).$$

In (H4), differentiability refers to the norm given by $\|(\alpha, \phi)\| = |\alpha| + \|\phi|_{C^2}$ on $\mathbb{R} \times C^2$. Notice that condition (H6) is weaker than continuity of the map

$$J \times U^* \ni (\alpha, \phi) \mapsto D_{2,e}^2 g^*(\alpha, \phi) \in L^2(C^1, \mathbb{R}^n),$$

where $L^2(C^1, \mathbb{R}^n)$ denotes the Banach space of continuous bilinear maps $C^1 \times C^1 \to \mathbb{R}^n$, with the appropriate norm.

Suppose $\phi^* \in U^* \subset C^2$ satisfies

$$g^*(\alpha, \phi^*) = 0 \text{ for all } \alpha \in J,$$

so that ϕ^* is a stationary point for all $\alpha \in J$.

For $\alpha \in J$, set $L(\alpha) = D_2 g(\alpha, \phi^*)$, and let $A(\alpha)$ denote the generator of the strongly continuous semigroup on C given by the IVP

$$y'(t) = D_{2,e} g(\alpha, \phi^*) y_t, \; y_0 = \chi \in C.$$

The spectral assumptions for the Hopf bifurcation theorem from [66] are the following.

(L1): There is a continuously differentiable map $\lambda : I \to \mathbb{C}$, $I \subset J$ an open interval, such that each $\lambda(\alpha)$, $\alpha \in I$, is a simple eigenvalue of $A(\alpha)$.

(L2): For some $\alpha_0 \in I$, $Re\, \lambda(\alpha_0) = 0$ and $\omega_0 = Im\, \lambda(\alpha_0) > 0$ and

$$\frac{d}{d\alpha}(Re\, \lambda)(\alpha_0) \neq 0.$$

(L3): For every integer $k \in \mathbb{Z} \setminus \{-1, 1\}$, $i\, k\, \omega_0$ is not an eigenvalue of $A(\alpha_0)$.

The following local Hopf bifurcation theorem is obtained in [66]:

Theorem 5.0.1. *Suppose (H1-H6) and (L1-L3) hold. Then there are an interval $M \subset \mathbb{R}$ with $0 \in M$ and continuously differentiable mappings $u^* : M \to C^1$, $\omega^* : M \to \mathbb{R}$ and $\alpha^* : M \to I$ with*

$$u^*(0) = \phi^*, \alpha^*(0) = \alpha_0, \omega^*(0) = \omega_0$$

such that for each $a \in M$ there is a periodic solution $x : \mathbb{R} \to \mathbb{R}^n$ of the equation

$$x'(t) = g(\alpha^*(a), x_t)$$

with $x_0 = u^*(a)$ and with period $\frac{\omega^*(a)}{2\pi}$.

To our knowledge, Theorem 5.0.1 is the first Hopf bifurcation result for differential equations with state-dependent delay. A related earlier result is due to H. L. Smith [196] who proved bifurcation of periodic solutions from a stationary point for a system of integral equations with state-dependent delay, by reduction to a Hopf bifurcation theorem of Hale and de Oliveira [97] for equations with time-invariant but parameter-dependent delay.

6. Differentiability of solutions with respect to parameters

6.1. **Preliminaries.** This section deals with nonautonomous parametrized state-dependent delay systems of the form

$$x'(t) = g\Big(t, x(t), x(t - \tau(t, x_t, \sigma)), \theta\Big), \qquad t \in [0, T], \tag{6.1.1}$$

with initial condition

$$x(t) = \phi(t), \qquad t \in [-h, 0]. \tag{6.1.2}$$

Here σ and θ are parameters in the delay function τ and in g belonging to normed linear spaces Σ and Θ, respectively. In the sequel we consider also the initial function ϕ in the IVP (6.1.1)-(6.1.2) as parameters, and denote the corresponding solution by $x(\cdot; \phi, \sigma, \theta)$, and its segment function at t by $x(\cdot; \phi, \sigma, \theta)_t$.

Suppose, e.g., a system of the form

$$y'(s) = \tilde{g}\Big(s, y(s), y(s - \tilde{\tau}(s, y_t))\Big), \qquad s \in [t_0, t_0 + T]$$

is given. Then introducing $t = s - t_0$ and $x(t) = y(t + t_0)$ we can transform the equation into the form (6.1.1) with $g(t, \psi, u, \theta) = \tilde{g}(t + \theta, \psi, u)$ with $\theta = t_0$, and $\tau(s, \psi, \sigma) = \tilde{\tau}(s + \sigma, \psi)$ with $\sigma = t_0$. In this case $\Sigma = \Theta = \mathbb{R}$. Of course, (6.1.1) contains more general cases as well, e.g., σ and θ can be coefficient functions in the delay function τ and g, respectively. In this case Σ and Θ will be infinite dimensional function spaces.

As we have seen in Section 3, in general the IVP (6.1.1)-(6.1.2) has a unique solution only if the initial function ϕ is Lipschitz continuous, or equivalently, if ϕ belongs to the Banach space $W^{1,\infty}$ of absolutely continuous functions $\phi : [-h, 0] \to \mathbb{R}^n$ with essentially bounded derivatives, with the norm defined by

$$|\phi|_{W^{1,\infty}} = \max\{|\phi|_C, \operatorname{ess\,sup}\{|\phi'(s)| \; : \; s \in [-h, 0]\}\}.$$

We define the parameter space for the IVP (6.1.1)-(6.1.2) as

$$\Gamma = W^{1,\infty} \times \Sigma \times \Theta,$$

and the norm on Γ by

$$|\gamma|_\Gamma = |(\phi, \sigma, \theta)|_\Gamma = |\phi|_{W^{1,\infty}} + |\sigma|_\Sigma + |\theta|_\Theta.$$

We assume throughout this section that $\Omega_1 \subset \mathbb{R}^n$, $\Omega_2 \subset \mathbb{R}^n$, $\Omega_3 \subset \Theta$, $\Omega_4 \subset C$ and $\Omega_5 \subset \Sigma$ are open subsets of the respective spaces, $T > 0$ is finite or $T = \infty$ (in the latter case $[0, T]$ means $[0, \infty)$), and

(D1) **(i):** $g : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n \times \Theta \supset [0, T] \times \Omega_1 \times \Omega_2 \times \Omega_3 \to \mathbb{R}^n$ is continuous; and

(ii): g is locally Lipschitz continuous with respect to its second, third and fourth variables in the following sense: For every $\alpha \in (0, T]$, for every compact subsets $M_i \subset \Omega_i$ $(i = 1, 2)$ of $\mathbb{R}^n$, and for every closed and bounded subset $M_3 \subset \Omega_3$ of Θ there exists $L_1 = L_1(\alpha, M_1, M_2, M_3)$ such that $|f(t, v, w, \theta) - f(t, \bar{v}, \bar{w}, \bar{\theta})| \le L_1\Big(|v - \bar{v}| + |w - \bar{w}| + |\theta - \bar{\theta}|_\Theta\Big)$, for $t \in [0, \alpha]$, $v, \bar{v} \in M_1$, $w, \bar{w} \in M_2$, and $\theta, \bar{\theta} \in M_3$;

(D2) **(i):** $\tau : \mathbb{R} \times C \times \Sigma \supset [0, T] \times \Omega_4 \times \Omega_5 \to \mathbb{R}$ is continuous, $0 \le \tau(t, \psi, \sigma) \le h$ for $t \in [0, T]$, $\psi \in \Omega_4$, and $\sigma \in \Omega_5$; and

(ii): $\tau(t,\psi,\sigma)$ is locally Lipschitz-continuous in ψ and σ in the following sense: For every $\alpha \in (0,T]$, for every compact subset $M_4 \subset \Omega_4$ of C and for every closed, bounded subset $M_5 \subset \Omega_5$ of Σ there exists a constant $L_2 = L_2(\alpha, M_4, M_5)$ such that $|\tau(t,\psi,\sigma) - \tau(t,\bar\psi,\bar\sigma)| \le L_2\Big(|\psi - \bar\psi|_C + |\sigma - \bar\sigma|_\Sigma\Big)$ for $t \in [0,\alpha]$, $\psi, \bar\psi \in M_4$, and $\sigma, \bar\sigma \in M_5$.

We, of course, assume that a parameter $\bar\gamma = (\bar\phi, \bar\sigma, \bar\theta) \in \Gamma$ satisfies the compatibility condition

$$\bar\phi(0) \in \Omega_1, \quad \bar\phi(-\tau(0,\bar\phi,\bar\sigma)) \in \Omega_2, \quad \bar\theta \in \Omega_3, \quad \bar\phi \in \Omega_4, \quad \text{and} \quad \bar\sigma \in \Omega_5. \tag{6.1.3}$$

It is known [58] that the IVP (6.1.1)-(6.1.2) has a unique solution for each parameter $(\bar\phi, \bar\sigma, \bar\theta) \in \Gamma$, moreover, the solution is Lipschitz continuous with respect to the parameters [101]. We denote the open ball with radius δ centered at $\bar\gamma$ in Γ by $G_\Gamma(\bar\gamma, \delta)$, i.e., $G_\Gamma(\bar\gamma,\delta) = \{\gamma \in \Gamma : |\gamma - \bar\gamma|_\Gamma < \delta\}$. The next result is proved, e.g., in [101].

Theorem 6.1.1. *Suppose (D1) (i), (ii), (D2) (i), (ii). For any $\bar\gamma = (\bar\phi, \bar\sigma, \bar\theta) \in \Gamma$ satisfying (6.1.3) there exist $\alpha > 0$, $\delta > 0$ and $L = L(\alpha, \bar\gamma, \delta)$ such that the IVP (6.1.1)-(6.1.2) has a unique solution on $[-h, \alpha]$ for any $\gamma \in G_\Gamma(\bar\gamma, \delta)$, and*

$$|x(\cdot;\gamma)_t - x(\cdot;\bar\gamma)_t|_{W^{1,\infty}} \le L|\gamma - \bar\gamma|_\Gamma \qquad \text{for } t \in [0,\alpha],\ \gamma \in G_\Gamma(\bar\gamma,\delta).$$

6.2. **Pointwise differentiability with respect to parameters.** In this subsection we study differentiability of the function $\gamma \mapsto x(t;\gamma)$ where t is fixed. In addition to (D1) (i), (ii) and (D2) (i), (ii) we need

(D1) (iii): $g : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n \times \Theta \supset [0,T] \times \Omega_1 \times \Omega_2 \times \Omega_3 \to \mathbb{R}^n$ is continuously differentiable with respect to its second, third and fourth variables;

(D2) (iii): $\tau : \mathbb{R} \times C \times \Sigma \supset [0,T] \times \Omega_4 \times \Omega_5 \to [0,\infty)$ is continuously differentiable with respect to its second and third variables.

As it was shown in Subsection 3.2, solutions corresponding to a parameter value from the parameter set

$$\Pi = \{(\phi,\sigma,\theta) \in \Gamma : \phi'(0) = g(0,\phi(0),\phi(-\tau(0,\phi,\sigma)),\theta), \quad \phi \in C^1\}$$

are continuously differentiable on $[-h,\alpha]$. The key point of the proof of the following result is the same as that of Theorem 3.2.1, namely, the differentiability of the evaluation map $Ev : C^1 \times [-h,0] \to \mathbb{R}^n$.

Theorem 6.2.1. *Suppose (D1) (i)–(iii) and (D2) (i)–(iii), $(\bar\phi,\bar\sigma,\bar\theta) \in \Pi$, and let $\delta > 0$ and $\alpha > 0$ be such that the IVP (6.1.1)-(6.1.2) has a unique solution on $[-h,\alpha]$ for any $\gamma \in G_\Gamma(\bar\gamma,\delta)$. Then for any $t \in [0,\alpha]$ the function*

$$x(t;\cdot) : \Gamma \supset G_\Gamma(\bar\gamma,\delta) \to \mathbb{R}^n$$

is differentiable at $\bar\gamma$, and its derivative is given by

$$D_2 x(t;\bar\gamma)u = z(t;\bar\gamma,u), \qquad u \in \Gamma,$$

where $z(\cdot;\bar{\gamma},u)$ is the solution of the time-dependent delay system

$$
\begin{aligned}
z'(t;\bar{\gamma},u) &= D_2g(t,\bar{x}(t),\bar{x}(t-\bar{r}(t)),\bar{\theta})z(t;\bar{\gamma},u)\\
&\quad + D_3g(t,\bar{x}(t),\bar{x}(t-\bar{r}(t)),\bar{\theta})\Big(-\bar{x}'(t-\bar{r}(t))D_2\tau(t,\bar{x}_t,\bar{\sigma})z(\cdot;\bar{\gamma},u)_t\\
&\quad + z(t-\bar{r}(t);\bar{\gamma},u)-\bar{x}'(t-\bar{r}(t))D_3\tau(t,\bar{x}_t,\bar{\sigma})u^{\sigma}\Big)\\
&\quad + D_4g(t,\bar{x}(t),\bar{x}(t-\bar{r}(t)),\bar{\theta})u^{\theta}, \qquad t\in[0,\alpha],\\
z(t;\bar{\gamma},u) &= u^{\phi}(t), \qquad t\in[-r,0],
\end{aligned}
$$

and where $\bar{x}(t)=x(t;\bar{\gamma})$, $\bar{r}(t)=\tau(t,\bar{x}_t,\bar{\sigma})$, and $u=(u^{\phi},u^{\sigma},u^{\theta})\in\Gamma$.

We refer to [101] for the proof of Theorem 6.2.1. Here we just make some remarks on the choice of the norm on Γ. It is clear that at some point in the proof of Theorem 6.2.1 it is necessary to be able to differentiate the composite function $F(t,\psi)=\psi(-\tau(t,\psi))$. Suppose $\psi\in C^1$ is fixed, and consider

$$
\begin{aligned}
F(t,\psi+u)-F(t,\psi) &= \psi(-\tau(t,\psi+u))+u(-\tau(t,\psi+u))-\psi(-\tau(t,\psi))\\
&= \psi'(-\tau(t,\psi))(-\tau(t,\psi+u)+\tau(t,\psi))+u(-\tau(t,\psi))\\
&\quad +\omega(t,\psi,u)+u(-\tau(t,\psi+u))-u(-\tau(t,\psi)),
\end{aligned}
$$

where

$$\omega(t,\psi,u)=\psi(-\tau(t,\psi+u))-\psi(-\tau(t,\psi))-\psi'(-\tau(t,\psi))(-\tau(t,\psi+u)+\tau(t,\psi)).$$

Therefore we expect that

$$D_2F(t,\psi)u=-\psi'(-\tau(t,\psi))D_2\tau(t,\psi)u+u(-\tau(t,\psi))$$

using an appropriate norm on the domain. The assumptions $\psi\in C^1$, (D2) (iii) and the chain rule combined imply immediately that $|\omega(t,\psi,u)|/|u|_C\to 0$ as $|u|_C\to 0$. But to control the term $u(-\tau(t,\psi+u))-u(-\tau(t,\psi))$ the C-norm is not suitable, we need the stronger $W^{1,\infty}$-norm: The Mean Value Theorem yields

$$\frac{|u(-\tau(t,\psi+u))-u(-\tau(t,\psi))|}{|u|_{W^{1,\infty}}}\le|\tau(t,\psi+u)-\tau(t,\psi)|,$$

and the right hand side of the preceding inequality tends to 0 as $|u|_{W^{1,\infty}}\to 0$, due to the continuity of τ. Therefore D_2F defined above is, in fact, the derivative of the function $F(t,\cdot)\,:\,W^{1,\infty}\to\mathbb{R}^n$ at $\psi\in C^1$ for any t.

6.3. **Differentiability with respect to parameters in norm.** Since the condition $\bar{\gamma}\in\Pi$ in the previous subsection may be inconvenient for certain applications, we explore different spaces to study differentiability in it for the case when the initial function and the solution segments are only $W^{1,\infty}$ functions.

First we introduce some notation and definitions. $W^{1,p}_{\alpha}$ ($1\le p<\infty$) denotes the Banach space of absolutely continuous functions $\psi\,:\,[-h,\alpha]\to\mathbb{R}^n$ of finite norm

$$|\psi|_{W^{1,p}_{\alpha}}=\left(\int_{-h}^{\alpha}|\psi(s)|^p+|\psi'(s)|^p\,ds\right)^{1/p}.$$

Similarly, $W^{1,\infty}_{\alpha}$ denotes the Banach space $W^{1,\infty}_{\alpha}([-h,\alpha],\mathbb{R}^n)$.

Consider a linear space Y and let $|\cdot|$ and $\|\cdot\|$ be two norms defined on Y. We say that $(Y,|\cdot|)$ is a quasi-Banach space with respect to the norm $\|\cdot\|$ if for any $r>0$ the set $\{y\in Y\,:\,\|y\|\le r\}$ is complete in the norm $|\cdot|$. See [98].

In addition to (D1) (i)–(iii) and (D2) (i)–(iii) we use in this subsection the following condition

(D2) **(iv):** $\tau : \mathbb{R} \times C \times \Sigma \supset [0,T] \times \Omega_4 \times \Omega_5 \to [0,\infty)$ is continuously differentiable with respect to its first variable; and

(v): $D_1\tau$, $D_2\tau$ and $D_3\tau$ are locally Lipschitz continuous in the following sense: For every $\alpha \in (0,T]$, for every compact subset $M_4 \subset \Omega_4$ of C and for every closed, bounded subset $M_5 \subset \Omega_5$ of Σ there exists a constant $L_3 = L_3(\alpha, M_4, M_5)$ such that

$$\begin{aligned}
&|D_1\tau(t,\psi,\sigma) - D_1\tau(t,\bar\psi,\bar\sigma)| \le L_3\Big(|\psi-\bar\psi|_C + |\sigma-\bar\sigma|_\Sigma\Big),\\
&\|D_2\tau(t,\psi,\sigma) - D_2\tau(t,\bar\psi,\bar\sigma)\|_{\mathcal{L}(C,\mathbb{R})} \le L_3\Big(|\psi-\bar\psi|_C + |\sigma-\bar\sigma|_\Sigma\Big),\\
&\|D_3\tau(t,\psi,\sigma) - D_3\tau(t,\bar\psi,\bar\sigma)\|_{\mathcal{L}(\Sigma,\mathbb{R})} \le L_3\Big(|\psi-\bar\psi|_C + |\sigma-\bar\sigma|_\Sigma\Big)
\end{aligned}$$

for $t \in [0,\alpha]$, $\psi, \bar\psi \in M_4$, and $\sigma, \bar\sigma \in M_5$.

Hale and Ladeira [98] investigated differentiability of solutions to the constant delay equation

$$x'(t) = f(x(t), x(t-\tau))$$

with respect to the delay, τ. They showed by means of an extension of the Uniform Contraction Principle to quasi-Banach spaces that the map

$$[0,h] \to W^{1,1}_\alpha, \quad \tau \mapsto x(\cdot;\tau)$$

is differentiable. Note that in their proof the integral norm of $W^{1,1}_\alpha$ played a crucial role; it can be replaced by a more general $W^{1,p}_\alpha$-norm, but not by the stronger $W^{1,\infty}_\alpha$-norm. This result suggests that the set $W^{1,\infty}_\alpha$ equipped with the norm $|\cdot|_{W^{1,p}_\alpha}$ could possibly be used as the state space for solutions. It might be a reasonable choice since (see e.g. [101]) the parameter map $(\phi,\theta,\sigma) \mapsto x(\cdot;\phi,\theta,\sigma)_t$ is Lipschitz continuous in both the $|\cdot|_{W^{1,\infty}}$ and $|\cdot|_{W^{1,p}}$-norms while the time map $t \mapsto x(\cdot;\phi,\theta,\sigma)_t$ is continuous only in the $|\cdot|_{W^{1,p}}$, but not in the $|\cdot|_{W^{1,\infty}}$-norm. This indicates that the set $W^{1,\infty}$ equipped with the $|\cdot|_{W^{1,p}}$-norm (which is not a Banach space, it is only a quasi-Banach space with respect to the $|\cdot|_{W^{1,\infty}}$-norm) could be considered as a "natural" state space for state-dependent delay equations.

We follow the usual procedure to study differentiability with respect to parameters: Introducing $y(t) = x(t) - \tilde\phi(t)$ where $\tilde\phi$ is the extension of ϕ to $[-h,\alpha]$ by $\tilde\phi(t) = \phi(0)$ for $0 < t \le \alpha$, we rewrite the IVP (6.1.1)-(6.1.2) as a fixed point equation $S(y,\phi,\theta,\sigma) = y$, with the operator S given by

$$S(y,\phi,\theta,\sigma)(t) = \begin{cases} 0, & t \in [-h,0]\\ \displaystyle\int_0^t g\Big(u,\, y(u)+\tilde\phi(u),\, \Lambda(u, y_u+\tilde\phi_u,\sigma),\theta\Big)\,du, & t\in[0,\alpha],\end{cases}$$

and with $\Lambda(t,\psi,\sigma) = \psi(-\tau(t,\psi,\sigma))$. In order to apply the Uniform Contraction Principle in this setting we need continuous differentiability of S with respect to y, ϕ, θ and σ in the $W^{1,p}_\alpha$ norm. It turns out that instead of the pointwise differentiability of Λ with respect to ψ and σ studied in the previous subsection it is enough to have the differentiability of the composite function $t \mapsto \Lambda(t,x_t,\sigma)$ with respect to x and σ in a norm of "L^p-type", for $x \in W^{1,\infty}_\alpha$.

Brokate and Colonius [29] studied equations of the form

$$x'(t) = f\Big(t, x(r(t,x(t)))\Big), \qquad t\in[a,b]$$

and investigated differentiability of the composition operator

$$A\ :\ W^{1,\infty}([a,b];\mathbb{R}) \supset \bar{X} \to L^p([a,b];\mathbb{R}), \qquad A(x)(t) = x(r(t,x(t))).$$

They assumed that r is twice continuously differentiable satisfying $a \le r(t,v) \le b$ for all $t \in [a,b]$ and $v \in \mathbb{R}$, and considered as domain of A the set

$$\bar{X} = \Big\{ x \in W^{1,\infty}([a,b];\mathbb{R}) \ :\ \text{There exists } \epsilon > 0 \text{ s.t. } \frac{d}{dt}\Big(r(t,x(t))\Big) \ge \epsilon \\ \text{for a.e. } t \in [a,b] \Big\}.$$

It was shown in [29] that under these assumptions A is continuously differentiable with the derivative given by

$$(DA(x)u)(t) = x'(r(t,x(t)))D_2 r(t,x(t))u(t) + u(r(t,x(t)))$$

for $u \in W^{1,\infty}([a,b],\mathbb{R})$.

Both the strong $W^{1,\infty}$ norm on the domain and the weak L^p norm on the range, together with the choice of the domain seemed to be necessary to obtain the results in [29]. Note that Manitius in [163] used a similar domain and norm when he studied linearization for a class of state-dependent delay systems.

Clearly, to apply the Uniform Contraction Principle to the operator S we have to use the same norm on the domain and range of S. It turns out that the following "product norm" preserves the essential properties of the different norms used in [98] and [29]: Let $x \in W^{1,\infty}_\alpha$, and decompose x as $x = y + \tilde{\phi}$, (where $\phi(t) = x(t)$ for $t \in [-r,0]$, and $\tilde{\phi}$ is the extension of ϕ to $[-r,\alpha]$ by $\tilde{\phi}(t) = \phi(0)$), and define the norm of x by

$$|x|_{X^{1,p}_\alpha} = \left(\int_0^\alpha |y'(u)|^p du\right)^{1/p} + |\phi|_{W^{1,\infty}},$$

and consider the normed linear space $X^{1,p}_\alpha \equiv (W^{1,\infty}_\alpha, |\cdot|_{X^{1,p}_\alpha})$. The norm $|\cdot|_{X^{1,p}_\alpha}$ is weaker than the $|\cdot|_{W^{1,\infty}_\alpha}$ norm, but stronger than the $|\cdot|_{W^{1,p}_\alpha}$ norm (see [108]).

This norm is "strong enough" that the methods of [29], with minor modifications, provide differentiability of the composition map

$$B\ :\ X^{1,p}_\alpha \times \Sigma \supset U_1 \times U_2 \to L^p([0,\alpha];\mathbb{R}^n), \qquad B(x,\sigma)(t) = \Lambda(t, x_t, \sigma),$$

on a suitable domain $U_1 \times U_2$. On the other hand, $|\cdot|_{X^{1,p}_\alpha}$ is "weak enough" that using the differentiability of the operator B above we can obtain obtain differentiability of the operator $S\ :\ X^{1,p}_\alpha \times W^{1,\infty} \times \Theta \times \Sigma \supset V_1 \times V_2 \times V_3 \times V_4 \to X^{1,p}_\alpha$ with respect to y, ϕ, θ and σ. Moreover it is possible to use a modification of the Uniform Contraction Principle to get differentiability of the fixed point (the solution of the IVP) with respect to the parameters ϕ, θ and σ in the $|\cdot|_{X^{1,p}_\alpha}$ norm. Since this product norm is stronger than the $|\cdot|_{W^{1,p}_\alpha}$ norm, the result implies the differentiability of solutions in the latter norm as well. For more details and the proof of the next result we refer to [108].

Theorem 6.3.1. *Suppose (D1) (i)–(iii) and (D2) (i)–(v), and let* $\bar{\delta} > 0$ *and* $\alpha > 0$ *be such that the IVP (6.1.1)-(6.1.2) has a unique solution on* $[-h,\alpha]$ *for any* $\gamma \in G_\Gamma(\bar{\gamma},\bar{\delta})$, *and suppose there exists* $\epsilon > 0$ *such that the solution* $\bar{x} = x(\cdot;\bar{\gamma})$ *satisfies*

$$\frac{d}{dt}\Big(t - \tau(t,\bar{x}_t,\bar{\sigma})\Big) \ge \epsilon \quad \textit{a.e. } t \in [0,\alpha].$$

Then there exists $\delta > 0$ such that the functions

$$\Gamma \supset G_\Gamma(\bar{\gamma}, \delta) \to X^{1,p}_\alpha, \qquad \gamma \mapsto x(\cdot;\gamma)$$

and

$$\Gamma \supset G_\Gamma(\bar{\gamma}, \delta) \to W^{1,p}_\alpha, \qquad \gamma \mapsto x(\cdot;\gamma)$$

are continuously differentiable.

An application of differentiability of solutions with respect to parameters was given in [102], where estimation of unknown parameters in state-dependent delay equations was studied. The goal of the work was to find a parameter value which minimizes a least square cost function $P(\gamma) = \sum_{k=1}^{N}(x(t_i;\gamma) - y_i)^2$, where y_i ($i = 1, \ldots, N$) are measurements of the solution at time points t_i ($i = 1, \ldots, N$). The so-called method of quasilinearization was adopted and numerically tested for state-dependent delay equations for cases where the parameters were infinite dimensional, e.g., the initial function. This algorithm is based on Newton's method and uses the derivative of P, hence also $D_2x(t;\gamma)$. The convergence of the estimation method was observed also for those cases where $D_2x(t;\gamma)$ did not exist in the pointwise sense of Theorem 6.2.1 but only in a norm, as stated in Theorem 6.3.1. For example, when the initial function is approximated by piecewise linear splines, then Theorem 6.2.1 is not applicable to solutions corresponding to such parameters since they belong to $W^{1,\infty}$ but not to C^1.

7. Periodic solutions via fixed point theory

Over the last 40 years the most general results on existence of periodic solutions to autonomous delay differential equations, with constant and also with state-dependent delay, have been obtained using topological fixed point theorems and the fixed point index. The first step in applying these tools is the construction of a *return map* : For initial data in a suitably chosen set K one follows the solution segments until they return to K. Fixed points of the return map define periodic solutions. The search for a suitable set K requires a priori knowledge about the desired periodic solutions and about their role in the global dynamics generated by the delay equation. Hypotheses of fixed point theorems must also be satisfied. This indicates that in general the search for K may be a nontrivial task. The finer (more restrictive) a structure a domain K of a return map has, the more qualitative information about the periodic solution is provided.

It is not uncommon that domains of return maps or their closures contain a known fixed point which also is a stationary point of the semiflow. Therefore, to obtain a *nonconstant* periodic solution, one needs to find another fixed point. While impossible in case the trivial fixed point is globally attracting, there is hope in case the trivial fixed point is unstable. A weak topological notion of instability, which proved very useful, is Browder's concept of *ejectivity* [30]. A fixed point x of a map $f : M \to N$, N a topological space and $M \subset N$, is called ejective if there exists a neighbourhood V of x in M so that for each $y \in V \setminus x$ there is $j \in \mathbb{N}$ with $f^j(y) \notin V$. Ejectivity was deeply explored and first applied by Nussbaum (see, e. g., [178, 179]), who also proved the first result on existence of periodic solutions for differential equations with state-dependent delay [178]. In the next subsection, Subsection 7.1, we describe a very general result of Mallet-Paret, Nussbaum, and Paraskevopoulos [159] on existence of periodic solutions and its proof, which is based on ejectivity.

The Subsection 7.2 deals with a model from Subsection 2.6 where the delay is governed by a differential equation which involves the state. This example was studied by Arino, Hadeler and Hbid [7] and Magal and Arino [151] and shows some of the specific difficulties caused by state-dependent delays in the search for periodic solutions.

Ejectivity is used also in the existence proofs in [4, 154, 138, 139].

Of course, ejectivity does not adequately reflect the unstable behaviour of solutions to decent differential equations. Close to unstable manifolds of equilibria, or in cones around such unstable manifolds, solutions to delay and other differential equations behave much more regular than expressed by ejectivity. Accordingly one can prove existence of periodic solutions and global bifurcation from stationary points also without recourse to ejectivity. Schauder's fixed point theorem and simple calculations of the fixed point index suffice if only unstable solution behaviour close to equilibria is exploited to a larger extent. This was done in [209, 210] and in Chapter XV of [55] for a class of RFDEs with constant delay. For equations with state-dependent delay, a proof of existence of periodic solutions along these lines has not yet been carried out.

Beyond existence and outside the scope of purely topological tools, uniqueness and stability of periodic orbits are of interest. In Subsection 7.3 we present results from [212, 215] where return maps are contractions or have a locally attracting fixed point, with the associated periodic orbit nontrivial, stable and hyperbolic.

The approach applies to single equations and systems with state-dependent delay where the nonlinearities are given by functions which do not vary much on long intervals.

7.1. **A general result by continuation.** In [159] Mallet-Paret, Nussbaum and Paraskevopoulos prove existence of periodic solutions for a rather general class of scalar RFDEs which include equations with state-dependent delay. They consider Equation (1.0.1) with $f : C \to \mathbb{R}$ continuous, $C = C([-h, 0], \mathbb{R})$, and assume that there are $\tau_0 \in (0, h]$ and a locally Lipschitz continuous function $g : \mathbb{R} \to \mathbb{R}$ satisfying the negative feedback condition

$$\xi\, g(\xi) < 0 \text{ for all } \xi \neq 0$$

so that

$$f(\phi) = g(\phi(-\tau_0))$$

for ϕ in the closed hyperplane H given by $\phi(0) = 0$. In particular, $f(0) = 0$, τ_0 is the delay on H, and the constant function zero is a solution to Equation (1.0.1). In Section 3 we mentioned the property (alL) from [159] of being *almost locally Lipschitzian*, to which condition (L) in Section 3 is closely related. Property (alL) requires local Lipschitz estimates for f which involve the norm on C but only arguments of f which are Lipschitz continuous. For f with property (alL) and at most linear growth Lipschitz continuous initial data $\phi \in C$ uniquely determine solutions $x : [-h, \infty) \to \mathbb{R}$ of Equation (1.0.1) with $x_0 = \phi$. Also bounds for solutions and continuous dependence on initial data are established. Under a more restrictive negative feedback condition, now for the functional f and involving also data $\phi \in C \setminus H$, it is shown that segments $x_t, t \geq 0$, of solutions which start from $x_0 = \phi$ in a closed bounded convex set

$$G^+ \subset \{\phi \in H : \phi(t) \geq 0 \text{ for all } t \in [-h, 0]\}$$

return to G^+ at a certain well-defined zero $z_2 = z_2(x_0) > 0$ provided the solution has at least one sign change on $(0, \infty)$ and a zero thereafter. On a subset $U^+ \subset G^+$ which is open with respect to the topology on G^+ induced by C the previous result yields a continuous return map $\Gamma_0 : U^+ \to G^+$. Actually, more is achieved here for later use: $\Gamma_0 = \Gamma(\cdot, 0)$, for a homotopy $\Gamma : U^+ \times [0, 1] \to G^+$ of modified return maps $\Gamma(\cdot, \alpha)$ associated with the members of a one-parameter family of RFDEs; the equation at $\alpha = 1$ has the property that the values of initial data on $[-h, -\tau_0)$ have no influence on the solution.

The extremal point $0 \in G^+$ of G^+ does not belong to U^+, and each fixed point of the return map Γ_0 defines a non-constant periodic solution of Equation (1.0.1). A fixed point exists if the fixed point index $i_{G^+}(\Gamma_0, U^+)$ of Γ_0 is defined and non-zero.

The remaining steps towards

$$i_{G^+}(\Gamma_0, U^+) \neq 0 \tag{7.1.1}$$

require some sort of linearization of the RFDE at the zero solution, in order to describe and exploit conditions for instability of the zero solution. Here the hypothesis (aFd) that f is *almost Fréchet differentiable* at 0 comes into play. It requires that the restriction of f to the space $C^{0,1} = C^{0,1}([-h, 0], \mathbb{R})$ of Lipschitz continuous data has a derivative at $0 \in C^1 \subset C^{0,1}$. The properties (alL) and (aFd) combined imply that this derivative $D(f|_{C^{0,1}})(0)$ extends to a continuous linear map $D_e(f|_{C^{0,1}})(0) : C \to \mathbb{R}$, like in conditions (S2) and (S3) from Section 3. A look back at the hypotheses on f shows that the recipe *freeze the delay at equilibrium,* *then linearize* mentioned in Section 3 is not sufficient to compute this extended derivative. The desired linear equation has the form

$$(7.1.2) \qquad \begin{aligned} v'(t) &= D_e(f|_{C^{0,1}})(0)v_t \\ &= -\beta\, v(t) - \gamma\, v(t-\tau_0) \end{aligned}$$

with constants $\beta \geq 0, \gamma \geq 0$, and is considered for initial data in the Banach space $C_{\tau_0} = C([-\tau_0, 0], \mathbb{R})$. On the cone

$$K = \{\phi \in C_{\tau_0} : \phi(t) \geq 0 \text{ for all } t \in [-\tau_0, 0],\ \phi(0) = 0\}$$

there is a return map $S_{\beta,\gamma} : K \to K$ similar to Γ_0, given by the solutions to Equation (7.2.2), with $S_{\beta,\gamma}(0) = 0$. In case the zero solution of Equation (7.2.2) is unstable the trivial fixed point $0 \in K$ has index zero;

$$(7.1.3) \qquad i_K(S_{\beta,\gamma}, 0) = 0.$$

To see this one can use ejectivity as in [178, 179], or follow [180], or proceed as in [210, 55].

The major steps in the proof of (7.2.1) are a reduction to

$$i_{G^+}(\Gamma_1, U^+) \neq 0$$

by homotopy invariance, and the deduction of the preceding inequality from (7.2.3). In this last step all basic properties normalization, additivity, homotopy invariance, and commutativity of the fixed point index are instrumental.

The skilful proof in [159] overcomes more obstacles than this brief outline indicates. The main theorem of [159] yields a wide variety of results on existence of periodic solutions for explicitly given RFDEs, in particular also for equations with multiple state-dependent delays of the form

$$x'(t) = G(x(t), x(t - r_1(x(t))), \ldots, x(t - r_m(x(t)))).$$

7.2. **Periodic solutions when delay is described by a differential equation.** As mentioned above, Nussbaum [178] used the ejective fixed point theorem to prove the existence of periodic solutions to the equation

$$x'(t) = -\alpha x(t - 1 - |x(t)|)(1 - x^2(t)).$$

Alt [4] and Kuang and Smith [138, 139] obtained periodic solutions for equations where the delay is given by a threshold condition.

While the major steps towards existence of so-called slowly oscillating periodic solutions remain essentially the same as for delay differential equations with constant delays, the technical details become more involved when delays are state-dependent. This refers both to the construction of the domain of a return map and to the verification that a trivial fixed point is ejective. To illustrate this we describe in the sequel work of Arino, Hadeler and Hbid [7] and of Magal and Arino [151] for the system

$$(7.2.1) \qquad \begin{cases} x'(t) = -f(x(t - r(t))), \\ r'(t) = q(x(t), r(t)). \end{cases}$$

Here the variation of the delay is determined by an ordinary differential equation. Standing assumptions are that the functions $f : \mathbb{R} \to [-M, M]$, $M > 0$, and $q : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are continuously differentiable, with

$$x\, f(x) > 0 \text{ for all } x \neq 0 \text{ and } f \text{ nondecreasing,}$$

and that for some fixed reals $r_2 > r_1 > 0$ and for all $x \in \mathbb{R}$,

$$q(x, r_2) < 0 < q(x, r_1).$$

Let $C = C([-r_2, 0], \mathbb{R})$. The hypotheses ensure that the IVP given by the system (7.2.1) and initial data $(\phi, r_0) \in C \times [r_1, r_2]$ with ϕ Lipschitz continuous has a unique solution $(x, r) = (x^{\phi, r_0}, r^{\phi, r_0})$, with x defined on $[-r_2, \infty)$ and r defined on $[0, \infty)$. The solution component x is Lipschitz continuous with $|x'(t)| \le M$ for all $t > 0$, and the hypothesis on q yields $r_1 < r(t) < r_2$ for all $t > 0$.

Incidentally, note that the system (7.2.1) can be reformulated as a special case of (1.0.1) with $n = 2, h = r_2, U = (\mathbb{R} \times [0, h])^{[-h,0]}$, and the corresponding functional being given by $(-f(\phi(-r_0)), q(\phi(0), r_0))$ for $(\phi, r_0) \in U$.

It also follows that the function $q(0, \cdot)$ has zeros in $[r_1, r_2]$. The additional hypothesis

$$\frac{\partial q}{\partial r}(0, r) < 0 \text{ on } [r_1, r_2]$$

implies that there is exactly one zero r_0^* of $q(0, \cdot)$ in $[r_1, r_2]$, and that the constant solution $r^* : t \mapsto r_0^*$ of the autonomous equation

$$r' = q(0, r)$$

is asymptotically stable. Below we shall see that this fact causes complications in view of ejectivity of a fixed point corresponding to the constant solution of system (7.2.1) given by $x(t) = 0$ and r^*.

In order to obtain that $t \mapsto t - r(t)$ is strictly increasing it is furthermore assumed that

$$q(x, r) < 1 \text{ on } \mathbb{R} \times [r_1, r_2].$$

The notion of a slowly oscillating solution, which is familiar from work on equations with constant delay $h > 0$ like, e. g.,

$$x'(t) = -f(x(t - h))$$

and means that zeros are isolated and spaced at distances larger than the delay h, is modified for the x-components of solutions to the system (7.2.1) according to the following definition: A function $x : [t_0, \infty) \to \mathbb{R}$, $t_0 \in \mathbb{R}$, is called slowly oscillating if its zeros form a disjoint union of closed intervals Z whose left endpoints have no accumulation point and are spaced at distances not less than r_2, with

$$\lim_{t \nearrow a} sign(x(t)) = - \lim_{s \searrow b} sign(x(s))$$

at each compact interval $Z = [a, b]$ with $t_0 < a$. Notice that the definition allows nonconstant functions which are zero on some unbounded interval $[t_1, \infty)$, $t_1 > t_0$, a phenomenon which occurs among the first components of solutions to the system (7.2.1).

The first step towards a return map is to show that certain initial data define solutions with slowly oscillating first component. In [7] it is shown that the first components of solutions with initial value component ϕ in the cone

$$\Gamma = \{\phi \in C([-r_2, 0]; \mathbb{R}) : \text{There exists } \theta \in [-r_2, -r_1] \text{ so that } \phi(\theta) = 0 \\ \text{and } \phi(s) < 0 \text{ for } s < \theta; 0 \le \phi(\theta) \le \phi(0) \text{ for } \theta \le s \le 0\}$$

return to Γ at a sequence of times whose distances are not less than r_2. More precisely, if (x, r) is a solution with initial value (ϕ, r_0) and $\pm\phi \in \Gamma$ then there are two sequences, possibly finite, of points t_i^* and zeros t_i of x in $[0,\infty)$ such that

$$t_0^* \geq 0,\ t_i^* + (r_2 - r_1) \leq t_{i+1} \leq t_{i+1}^* - r_1$$

and $\pm(-1)^{i+1}x$ is nondecreasing on the interval $[t_i^*, t_{i+1}^*]$. In particular, $\pm(-1)^i x_{t_i^*} \in \Gamma$ for each index $i > 0$. To reach the conclusion, it is required that the delay variation satisfies the smallness condition $(r_2 - r_1)|f(x)| < |x|$ for $x \neq 0$.

Uniqueness with respect to initial data is needed, and the domain for a return map should be convex. Therefore Γ must be modified. The smaller set

$$\Gamma_1 = \{\phi \in \Gamma : \phi \text{ is Lipschitzian and nondecreasing}\}$$

is convex, and closed in $C^{0,1}([-r_2, 0], \mathbb{R})$. Let $E = \Gamma_1 \times [r_1, r_2]$. For each integer $j \geq 1$, it is now natural to introduce operators P_j and P_j^+ on the space E by

$$\begin{aligned} P_j(\phi, r_0) &= (x_{t_j^*}, r(t_j^*)), \\ P_j^+(\phi, r_0) &= ((-1)^j x_{t_j^*}, r(t_j^*)). \end{aligned}$$

Obviously, the existence of $P_j(\phi, r_0)$ is subject to the condition that $t_1^*, t_2^*, \cdots, t_{j-1}^*$ exist. If this condition is satisfied but the solution starting from $(x_{t_{j-1}^*}, r(t_{j-1}^*))$ does not cross zero, then it is proved in [7] that $x(t) \to 0$ and $r(t) \to r^*$ as $t \to \infty$. In this case, one defines $P_j(\phi, r_0) = (0, r^*)$.

P_1 sends bounded sets of E into bounded sets of the product space $C([-r_2, 0], \mathbb{R}) \times [r_1, r_2]$, and P_1^+ is a compact and continuous operator from E into itself.

It is important to know when a first positive zero $t_1 = t_1(\phi, r_0)$ exists: This property is ensured for every $(\phi, r_0) \in (\Gamma_1 \cup (-\Gamma_1)) \times [r_1, r_2]$ if there exist $M > 0$ and $R_0 > 0$ with $M(r_2 - r_1) \leq R_0$, $|f(x)| \leq M$ for $x \in \mathbb{R}$ and $(2r_1 - r_2)|f(x) \geq |x|$ for $|x| \leq R_0$.

Unfortunately, the fixed point $(0, r^*)$ of any return map defined in a set containing $\{0\} \times [r_1, r_2]$ is not ejective. This follows immediately from positive invariance of the set $\{0\} \times [r_1, r_2]$ and asymptotic stability of the solution r^* to the equation $r' = q(0, r)$.

In order to obtain ejectivity smaller convex subsets

$$E_K = \{(\phi, r_0) \in E : |r_0 - r_0^*| \leq K\|\phi\|_C, \|\phi\|_C = |\phi(0)|\}$$

for $K > 0$ are introduced which contain from the obstacle for ejectivity $\{0\} \times [r_1, r_2]$ only the point $(0, r_0^*)$. For $R > 0$ and $K > 0$ let

$$E_{R,K} = \{(\phi, r_0) \in E : \|\phi\|_C \leq R, |r_0 - r_0^*| \leq K\|\phi\|_C\},$$

and define

$$\Gamma_2 = \{\phi \in \Gamma_1 : \phi(0) = \sup_{-r_2 \leq s \leq 0} |\phi(s)|\}.$$

In order for P_1^+ to map $(\Gamma_2 \times [r_1, r_2]) \cap E_{R,K}$ into E_K, a restriction on r_1 is needed. In [7] it is shown that for each $R > 0$ there exist $\tilde{r}_1 > 0$ and $K > 0$ such that for each $r_1 > \tilde{r}_1$ the operator P_1^+ maps $(\Gamma_2 \times [r_1, r_2]) \cap E_{R,K}$ into E_K. The key to the proof of the above statement is that for each $R > 0$ there exists $C(r_1) \in (0, 1)$ such that

$$\|x_{t_1^*}^{\phi, r_0}\|_C \geq C(r_1)\|\phi\|_C \tag{7.2.2}$$

for each $(\phi, r_0) \in \Gamma_2 \times [r_1, r_2]$ with $0 < \|\phi\|_C \leq R$. Estimates of this type exclude superexponential decay of slowly oscillating solutions and play an important role in work on the global dynamics of equations with constant delay.

The map $(\phi, r_0) \mapsto -x_{t_1^*}^{\phi,r_0}$ sends $E = \Gamma_1 \times [r_1, r_2]$ into Γ_2. The fact that $t_1 = t_1(\phi, r_0)$ is a zero of $x = x^{\phi,r_0}$ and integration of the first equation in (7.2.1) yield the estimate

$$\|x_{t_1^*}^{\phi,r_0}\|_C \le Mr_2,$$

for each $(\phi, r_0) \in E$. With this preparation, we can define $R = Mr_2$ and introduce the set

$$X = \{\phi \in \Gamma_2 : \|\phi\|_C \le R, \text{ess sup}_{-r_2 \le s \le 0}|\phi'(s)| \le M\}.$$

Choose $K > 0$ and $\tilde{r}_1 > 0$ as above, assume $r_1 > \tilde{r}_1$, and set

$$Y = (X \times [r_1, r_2]) \cap E_{R,K}.$$

Y is closed and convex, and the iterate P_1^2 defines a map $P : Y \to Y$.

It remains to show that $(0, r_0^*)$ is an ejective fixed point of P. Ejectivity of $(0, r_0^*)$ is a consequence of instability of the constant solution $t \mapsto (0, r_0^*)$ to the system (7.2.1), which in turn follows from instability for the linearized system. In [7] the technique *freeze the delay at equilibrium, then linearize* mentioned in Subsection 3.4 is applied to the first equation in (7.2.1) and yields

$$y'(t) = -y(t - r_0^*), \tag{7.2.3}$$

which is unstable for $r_0^* > \frac{\pi}{2}$ when f normalized so that $f'(0) = 1$. More precisely, the eigenvalues of the generator of the semigroup generated by Equation (7.2.2) on the space

$$C([-r_0^*, 0], \mathbb{R})$$

with largest real part are a complex conjugate pair $u \pm iv$ in the open right halfplane, and the associated realified generalized eigenspace U is 2-dimensional and consists of segments of solutions

$$\mathbb{R} \ni t \mapsto e^{ut}(a \cos(vt) + b \sin(vt))$$

to Equation (7.2.3). For $(a, b) \neq (0, 0)$ these solutions are slowly oscillating, due to $|v| < \frac{\pi}{r_0^*}$. The problem is now to transfer such unstable solution behaviour to the slowly oscillating solution components x of solutions (x, r) to the nonlinear system (7.2.1) which start from small initial data $(\phi, \rho) \in Y$ in the other state space $C([-r_2, 0], \mathbb{R}) \times [r_1, r_2]$. The proof in [7] proceeds by contradiction. Ejectivity means that there exists $\epsilon > 0$ so that for each $(\phi, r_0) \in Y$ with $0 < \|\phi\|_C \le \epsilon$ there is an integer $j > 0$ such that $(x, r) = (x^{\phi,r_0}, r^{\phi,r_0})$ and $t_j^* = t_j^*(\phi, r_0)$ satisfy $\|x_{t_j^*}\|_C \ge \epsilon$ or $|r(t_j^*) - r_0^*| \ge \epsilon$. It can be shown that in case the previous statement on ejectivity is not true, then there exists a constant $d > 0$ so that for all $(\phi, r_0) \in Y$ with $\|\phi\|_C \le \epsilon$ and for all $t \ge 0$ we have $\|x_t^{\phi,r_0}\|_C \le d\epsilon$ and $|r(t) - r_0^*| \le d\epsilon$.

The next step is to show that the spectral projection Π_U onto the realified generalized eigenspace U associated with $u \pm v$ in $C([-r_0^*, 0], \mathbb{R})$ satisfies

$$\gamma := \inf_{\|\phi\|=1, \phi \in \tilde{\Gamma}_2} \|\Pi_U \phi\| > 0, \tag{7.2.4}$$

under a further restriction about the variation of the delay

$$r_2 - r_1 < \delta \tag{7.2.5}$$

for some constant $\delta > 0$ (explicitly given in [7]) that depends on r_0^* only. Here and in what follows, $\tilde{\Gamma}_2$ is defined in a similar fashion as Γ_2, except we replace Γ_1 and Γ by $\tilde{\Gamma}_1$ and $\tilde{\Gamma}$ respectively, where the domain of ϕ is $[-r_0^*, 0]$ rather than $[-r_2, 0]$.

Let $\sup_{j\in\mathbb{N}} \|x_{t_j^*}\phi, r_0\|_C = \tilde{\epsilon} \le \epsilon$. We can choose an integer $j_0 > 0$ so that $\|x_{t_{j_0}^*}\phi, r_0\|_C \ge C\tilde{\epsilon}$ and thus, $|y(t_{j_0}^*)| \ge C\tilde{\epsilon}\gamma$, where $y(t) = \Pi_U x_t$, C and γ are given in (7.2.2) and (7.2.4). Note that the first equation of the system (7.2.1) can be written as $x'(t) = -x(t - r^*) + o(\tilde{\epsilon})$ for large t, by using the linearization and the fact that $|x'(t)| \le M$ and $|r(t) - r_0^*| \le d\tilde{\epsilon}$ for all $t \ge 0$.

Therefore,

$$y'(t) = A_U y(t) + o(\tilde{\epsilon})$$

with $A_U = \begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix}$. It then follows that

$$\frac{d}{dt}|y(t)| = \alpha|y(t)| + o(\tilde{\epsilon}).$$

Therefore,

$$|y(t)| = e^{\alpha(t-\sigma)}[|y(\sigma)| + \int_\sigma^t e^{\alpha(\sigma-s)}o(\tilde{\epsilon})ds],$$

which implies

$$|y(t)| \ge e^{\alpha(t-t_{j_0}^*)}[C\tilde{\epsilon}\gamma - \frac{o(\tilde{\epsilon})}{\alpha}] \ge e^{\alpha(t-t_{j_0}^*)}\frac{C\gamma}{2}\tilde{\epsilon}$$

if $\tilde{\epsilon}$ is small. This leads to a contradiction since $y(t) = \Pi_U x_t$ should be bounded by a constant multiple of $\tilde{\epsilon}$ for all $t \ge 0$.

In summary, under a few technical conditions including the negative feedback condition on the state variable x and the delay r, the smallness of the delay variation and an instability condition for an associated linear equation, Arino, Hadeler and Hbid obtain in [7] the existence of periodic solution with slowly oscillating first component.

In [151] Magal and Arino obtain such periodic solutions under weaker conditions on the delay variation - (7.2.5) is no longer needed - by means of a different argument which uses a modification of ejectivity and employs other cones of initial data. Magal and Arino consider a cone that was already used by Kuang and Smith in [138, 139], namely,

$$\begin{aligned} E_{KS} = \{(\phi, r_0) \in C^{0,1}([-r_2, 0], \mathbb{R}) \times [r_1, r_2] : \phi(-r_0) = 0 \\ \text{and } \phi \text{ nonincreasing on } [-r_0, 0]\}. \end{aligned}$$

Using arguments similar to those in [7] one can show that for $\epsilon = \pm 1$ and $(\epsilon\phi, r_0) \in E_{KS}$ there are reals $t_i^* = t_i^*(\phi, r_0)$ and zeros $t_i = t_i(\phi, r_0)$ of $x = x^{\phi, r_0}$ such that $t_0 = -r_0$, $t_0^* = 0$, $t_i^* \le t_{i+1}$, $t_i = t_i^* - r(t_i^*)$ for integers $i \ge 0$, and $\epsilon(-1)^{i+1}x(t)$ is nonincreasing on $[t_i^*, t_{i+1}^*]$, $x(t_i) = 0$, and $x(t_i^*) \ne 0$ if $\phi(0) \ne 0$. So, $(\epsilon(-1)^{i+1}x_{t_i^*}, r(t_i^*)) \in E_{KS}$. Let $X_0 = C^1([-r_2, 0], \mathbb{R}) \times [r_1, r_2]$, introduce

$$E_0 = \{(\phi, r_0) \in X_0 : \phi'(s) \ge 0 \text{ for } s \in [-r_0, 0], \phi(-r_0) = 0, \phi'(0) = 0\},$$

and define the return maps P_j and P_j^+ on E_0 exactly as in [7]. For each integer $j > 0$ one finds $P_{2j}(E_0) \subset E_0$.

The fact that for initial data $(\phi, r_0) \in C^1([-r_2, 0], \mathbb{R}) \times [r_1, r_2]$ the map $t \mapsto t - r^{\phi, r_0}(t)$ is increasing implies that on $[0, \infty)$ the solution $(x^{\phi, r_0}, r^{\phi, r_0})$ does not depend on the restriction of ϕ to $[-r_2, -r_0)$. The preceding observation suggests to consider initial data $(\tilde{\phi}, r_0)$ with $\tilde{\phi}$ defined only on $[-r_0, 0]$, and secondly, to modify the fixed point problem by transforming the initial delay r_0 to 1. Let

$$X_1 = C^1([-1, 0], \mathbb{R}) \times [r_1, r_2].$$

Magal and Arino introduce maps

$$L : X_0 \to X_1 \text{ and } Q : X_1 \to X_0$$

by

$$L(\phi, r_0) = (\psi, r_0),\ \psi(s) = \phi(s\, r_0)$$

and

$$Q(\psi, r_0) = (\phi, r_0),\ \phi(s) = \psi(s/r_0) \text{ on } [-r_0, 0],$$

$$\phi(s) = \frac{\psi'(-1)}{r_0}(s - r_0) \text{ on } [-r_2, -r_0).$$

Let E_1 denote the analogue of E_0 where X_0 and r_0 are replaced by X_1 and 1, respectively. E_1 is closed and convex, and $Q(E_1) \subset E_0$, $L(E_0) \subset E_1$. The maps $F_{2j} = L \circ P_{2j} \circ (Q|E_1)$ send E_1 into E_1. We have $F_2(0, r_0^*) = (0, r_0^*)$, and fixed points (ϕ, r_0) of F_2 with $\phi \neq 0$ yield periodic solutions of the system (7.2.1) with slowly oscillating first component. However, the second component of $F_2 : E_1 \to C^1([-1, 0], \mathbb{R}) \times [r_1, r_2]$ is not continuous at points $(0, r_0)$ with $r_0 \neq r_0^*$, which requires a further modification. The map $\tilde{F}_2 : E_1 \to E_1$ resulting from this is continuous and compact (with respect to the topology on $C^1([-1, 0], \mathbb{R})$) and retains the property that nontrivial fixed points define periodic solutions to (7.2.1) with slowly oscillating first component. It is shown in [151] that

(i) $\tilde{F}_2(0, r_0^*) = (0, r_0^*)$,
(ii) $\tilde{F}_2(\{0\} \times [r_1, r_2]) \subset \{0\} \times [r_1, r_2]$,
and
(iii) for every $\epsilon > 0$ there exist $c > 0$ and $\gamma \in [0, 1)$ with

$$|(\tilde{F}_1^{2j})_2(\phi, r_0) - r_0^*| \leq \gamma |r_0 - r_0^*|$$

for all $(\phi, r_0) \in E_1$ with

$$\|\phi\|_{C^1([-1,0],\mathbb{R})} + |r_0 - r_0^*| \leq \epsilon$$

and

$$\|\phi\|_{C^1([-1,0],\mathbb{R})} \leq c|r_0 - r_0^*|,$$

and for all integers $j \geq 1$.

Here the index 2 denotes the second component of the map $\tilde{F}_2^{2j}$. Properties (ii) and (iii) combined exclude that the fixed point $(0, r^*)$ of $\tilde{F}_2$ is ejective. Now the modification of ejectivity comes into play. Let X be a Banach space, $A \subset Y \subset X$, and assume that $g : Y \to Y$ has a fixed point $x_0 \in \partial_Y A$. Then x_0 is called semi-ejective on $Y \setminus A$ if there is a neighbourhood V of x_0 in Y so that for each $y \in V \setminus A$ there is an integer $m \geq 1$ with

$$g^m(y) \in Y \setminus V.$$

Arguments similar to those in [7] which exploit the instability of Equation (7.2.3) yield in [151] that the fixed point $(0, r_0^*)$ is semi-ejective on $E_1 \setminus (\{0\} \times [r_1, r_2])$. Finally, an extension of the ejective fixed point theorem in [151] to the case of semi-ejective fixed points guarantees that $\tilde{F}_2$ has a fixed point (ϕ, r_0) with $\phi \neq 0$, which defines the desired periodic solution.

7.3. **Attracting periodic orbits.** In [212] the system (2.2.1-2)

$$\begin{aligned} x'(t) &= v\left(\frac{c}{2}s(t-r)-w\right) \\ c\,s &= |x(t-s)+w|+|x(t)+w|, \end{aligned}$$

with positive parameters c, r, w, is studied. The function $v:\mathbb{R}\to\mathbb{R}$ is assumed to satisfy a negative feedback condition

$$\delta\, v(\delta)<0 \text{ for all } \delta\neq 0.$$

For continuous and bounded v the system defines a continuous semiflow on a set O which is open in a compact set $M\subset C=C([-h,0],\mathbb{R})$, with $h=\frac{4w}{c}+r$. The set M consists of Lipschitz continuous functions. The set O contains a closed subset K of initial data which define solutions whose segments x_t return to K, after an excursion into the ambient space O. This yields a continuous return map, on a domain without stationary points. If v is Lipschitz continuous and close to constants in $(-\infty,-\beta]$ and $[\beta,\infty)$, respectively, with $\beta>0$ sufficiently small, then one can estimate the Lipschitz constant of the return map in terms of Lipschitz constants for v and for its restrictions $v|_{(-\infty,-\beta]}$ and $v|_{[\beta,\infty)}$; under suitable further assumptions the return map becomes a contraction. The unique fixed point of the contracting return map belongs to a periodic orbit of the system (2.2.1-2) which is stable and exponentially attracting with asymptotic phase.

The observation which led to the method used in [212] and in earlier work on equations with constant delays is the following: If the function $g:\mathbb{R}\to\mathbb{R}$ in the equation

$$y'(t)=g(y(t-1)) \tag{7.3.1}$$

is constant on some interval I and if y remains long enough in I, say, for $t_0-1\leq t\leq t_0$, then for $t\geq t_0$ the solution y depends only on $y(t_0)$ and $g(I)$. This can be used to design simple-looking nonlinearities g, representing negative feedback, for which periodic solutions of Equation (7.3.1) can be computed explicitly, see, e. g., Chapter XV in [55]. Moreover, solutions which start from initial data close to the periodic orbit eventually merge into it. This is an extremely strong kind of orbital stability, giving hope that also for nonlinearities which are only close to constants on some intervals attracting periodic orbits may exist. If instead of the scalar Equation (7.3.1) more generally systems are considered then suitable nonlinearities which are constant on nontrivial intervals yield low-dimensional subsets of the state space which are positively invariant under the semiflow and absorb flowlines from a neighbourhood.

In [215] the system

$$u' = v \tag{7.3.2}$$

$$v' = -r\,v + A(p) \tag{7.3.3}$$

$$p = \frac{c}{2}\,s - w \tag{7.3.4}$$

$$c\,s = u(t-s)+u(t)+2w \tag{7.3.5}$$

is studied. In contrast to the system (2.2.1-2) the model (7.3.2-5) for position control by echo is now based on Newton's law. Instead of the constant time lag $r>0$ in Equation (2.2.1) there is now a friction term $-r\,v$ in Equation (7.3.3).

For suitable positive values of the parameters w, c, r and for certain functions $A:\mathbb{R}\to\mathbb{R}$ which represent negative feedback and are constant on $(-\infty,-\beta]$ and on $[\beta, \infty)$, with $\beta > 0$ small, the existence of a hyperbolic stable periodic orbit is established. The proof begins with a reformulation of the system as an equation of the form (1.0.1), with a continuously differentiable functional f defined on an open subset U of the space $C^1 = C^1([-h, 0], \mathbb{R}^2)$, for suitable $h > 0$. Several steps of the proof rely on the smoothness properties of the semiflow F on the solution manifold $X_f \subset U$ which are provided by Theorem 3.2.1. In X_f a thin, infinite-dimensional set I of initial data ϕ is found to which the flowlines $F(\cdot, \phi)$ return, after a journey through the ambient part of the manifold. The associated return map is not necessarily compact, which precludes an immediate application of Schauder's theorem in order to find a fixed point - not to speak of an attracting fixed point. But the return map is semiconjugate to an interval map which is differentiable. Estimates of derivatives of the interval map yield a unique, attracting fixed point of the latter, which can be lifted to the return map. The proof that the resulting periodic orbit of the system (7.3.2-5) is hyperbolic and stable involves a continuously differentiable Poincaré return map, on a hyperplane transversal to the periodic orbit, in addition to the previous return map on the thin set $I \subset X_f$, and a discussion of derivatives of iterates.

8. Attractors, singular perturbation, small delay, generic convergence, stability and oscillation

This section deals with limiting behaviour. Subsection 8.1 is concerned with long term dynamics and reports about the structure of a global attractor [134]. Subsection 8.2 describes work of Mallet-Paret and Nussbaum [154, 155, 156, 157, 158] about the asymptotic shape of periodic solutions when a parameter becomes large, Subsection 8.3 sketches an approach of Ouifki and Hbid [182] to existence of periodic solutions when delays are small, Subsection 8.4 reports about work of Bartha [21] on generic convergence of solutions, and Subsection 8.5 comments on further results about stability and oscillatory solution behaviour.

8.1. **An attracting disk.** The paper [134] studies the equation

$$x'(t) = -\mu x(t) + f(x(t - r(x(t)))) \tag{8.1.1}$$

with $\mu > 0$, $f \in C^2(\mathbb{R}, \mathbb{R})$, $f(0) = 0$, $f'(u) < 0$ for all $u \in \mathbb{R}$, $r \in C^1(\mathbb{R}, \mathbb{R})$, $r(0) = 1$, and $\sup_{u \in \mathbb{R}} f(u) < \infty$ provided $r(u) \geq 0$ for all $u \in \mathbb{R}$. The case $\mu = 0$ can also be handled with a slight modification. Then the delayed logistic equation (or Wright's equation)

$$y'(t) = -\alpha y(t - r(y(t)))[1 + y(t)]$$

with state-dependent delay and solutions satisfying $y(t) > -1$ is a particular case. Indeed, after the transformation $x = \log(1 + y)$ we obtain

$$x'(t) = -\alpha \left[e^{x\left(t - r\left(e^{x(t)} - 1\right)\right)} - 1 \right].$$

The aim is to describe the asymptotic behaviour of the slowly oscillatory solutions of Equation (8.1.1). Here a solution x of (8.1.1) is called slowly oscillatory if $|z' - z| > r(0) = 1$ for every pair of zeros z', z of x. The results are in part analogous to those of Walther [211] for the constant delay case $r \equiv 1$.

Let I_r denote the largest subinterval of $\mathbb{R}$ with $0 \in I_r$ and $r(u) \geq 0$ for all $u \in I_r$. First it is shown that for every element ϕ of the space $BC((-\infty, 0], I_r)$ of bounded continuous functions from $(-\infty, 0]$ into I_r, there is a solution $x : \mathbb{R} \to \mathbb{R}$ of Equation (8.1.1) through ϕ, i.e., x is continuous on $\mathbb{R}$, continuously differentiable on $(0, \infty)$, (8.1.1) holds for all $t > 0$, and $x|_{(-\infty,0]} = \phi$. If ϕ is Lipschitz continuous then x is unique.

In the next step four positive constants A, B, R, K are constructed such that

$$r([-B, A]) \subset (0, R],$$
$$\max_{(u,v) \in [-B,A] \times [-B,A]} | - \mu u + f(v)| \leq K,$$

moreover, for any solution $x : \mathbb{R} \to \mathbb{R}$ of (8.1.1) with $x|_{(-\infty,0]} \in BC((-\infty, 0], I_r)$ there exists $T \geq 0$ such that

$$x(t) \in [-B, A] \qquad \text{for all } t \geq T.$$

Therefore, from the point of view of the asymptotic ($t \to \infty$) behaviour, only those solutions are interesting which have values in $[-B, A]$.

Let C_R denote the space $C([-R, 0], \mathbb{R})$ equipped with the supremum norm. The set

$$L_K = \left\{ \phi \in C_R : \phi([-R, 0]) \subset [-B, A], \quad \sup_{-R \leq s < t \leq 0} \frac{|\phi(t) - \phi(s)|}{t - s} \leq K \right\}$$

is a compact convex subset of C_R. For every $\phi \in L_K$, Equation (8.1.1) has a unique solution $x^\phi : [-R, \infty) \to \mathbb{R}$ with $x|_{[-R,0]} = \phi$ and $x(t) \in [-B, A]$ for all $t \geq 0$. Then the relations

$$F(t, \phi) = x_t^\phi, \ t \geq 0, \ x_t^\phi(s) = x^\phi(t+s), \ -R \leq s \leq 0,$$

define a continuous semiflow F on L_K. In the sequel, only those solutions of (8.1.1) are considered whose segments are in L_K.

Define the compact subset

$$S = \{\phi \in L_K : \mathrm{sch}(\phi, [t-1, t]) \leq 1 \text{ for all } t \in [-R+1, 0]\}$$

of L_K, where $\mathrm{sch}(\phi, [t-1, t])$ denotes the number of sign changes of ϕ on the interval $[t-1, t]$. If x is a slowly oscillatory solution of (8.1.1), then all segments x_t belong to S.

The set S is positively invariant under the semiflow F. The restriction of F to $[0, \infty) \times S$ defines the continuous semiflow F_S on the compact metric space S. The global attractor $\mathcal{A}$ of F_S has the following properties:

(i) $\mathcal{A}$ is a compact connected subset of $S \subset L_K$.
(ii) For each $\phi \in \mathcal{A}$ there is a unique solution of (8.1.1) through ϕ on $\mathbb{R}$, which is also denoted by x^ϕ. The map $F_{\mathcal{A}} : \mathbb{R} \times \mathcal{A} \ni (t, \phi) \mapsto x_t^\phi \in \mathcal{A}$ is a continuous flow.
(iii) $\mathcal{A}$ is the union of $0 \in C_R$ and the segments x_t of the globally defined slowly oscillating solutions $x : \mathbb{R} \to [-B, A]$ of Equation (8.1.1).

The first main result is that a Poincaré–Bendixson type theorem holds on $\mathcal{A}$: The α- and ω-limit sets of phase curves in $\mathcal{A}$ are either $\{0\}$ or periodic orbits given by slowly oscillating periodic solutions. The second main result is that in case $\mathcal{A} \neq \{0\}$, the set $\mathcal{A}$ is homeomorphic to the 2-dimensional closed unit disk so that the unit circle corresponds to a periodic orbit given by a slowly oscillating periodic solution.

Below we list some of the technical tools used in the proofs.

There is an additional assumption on the delay function r: either

$$|r'(u)| < \frac{1}{K}, \qquad u \in [-B, A],$$

or

$$r \in C^2([-B, A], \mathbb{R}), \text{ and there is } a \in (0, 1) \text{ with } r''(u) \leq a\mu[r'(u)]^2, \ u \in [-B, A].$$

This assumption and the fact that the dependence of the delay on the state is of the simple form $r(x(t))$ seem to be crucial in several parts of the proof.

An important consequence of the above hypothesis on r is that the function

$$t \mapsto t - r(x(t))$$

is strictly increasing for solutions of (8.1.1). Another important fact is that for a suitable weighted difference $v : \mathbb{R} \to \mathbb{R}$ of two solutions x and y on $\mathbb{R}$, an equation of the form

$$v'(t) = \alpha(t) v(t - r(x(t))) \tag{8.1.2}$$

holds on $\mathbb{R}$ with a negative, bounded and continuous α. The backward uniqueness of solutions is a corollary.

A modified version of the well-known discrete Lyapunov functional of Mallet-Paret and Sell [153, 160, 161] is also introduced. Instead of on intervals with fixed length, the sign changes are counted for a solution x on intervals of the form $[t - r(x(t)), t]$. The properties are completely analogous to those of the constant delay case. In particular, this functional can be used to exclude the existence of solutions decaying to zero at ∞ or $-\infty$ faster than any exponential. By applying the discrete Lyapunov functional to a weighted difference of two solutions satisfying Equation (8.1.2), the number of sign changes of the difference can be controlled.

A return map P on the compact convex set

$$U = \{\phi \in L_K : \phi(0) = 0,\ \phi(s) \geq 0 \text{ for all } s \in [-1, 0]\}$$

is defined by $P(\phi) = x^{\phi}_{z_2}$, where at z_2 the second sign change of x^{ϕ} in $(0, \infty)$ occurs, and $P(\phi) = 0$ if there is at most one sign change in $(0, \infty)$. P is not necessarily continuous on U. However, $P|_{\mathcal{A} \cap U}$ is continuous. In addition, P restricted to $\{\phi \in \mathcal{A} \cap U : P(\phi) \neq 0\}$ is a homeomorphism onto $\mathcal{A} \cap U \setminus \{0\}$. It is also an essential step that $\mathcal{A} \cap U$ is connected. The elements of $\mathcal{A} \cap U \setminus \{0\}$ are exactly those segments x_s of globally defined slowly oscillating solutions $x : \mathbb{R} \to [-B, A]$ for which $x(s) = 0$ and $x'(s) < 0$.

An asymptotic expansion for slowly oscillating solutions converging to zero as $t \to -\infty$ is also proved. It relates solutions of Equation (8.1.1) to solutions of the associated linear equation

$$y'(t) = -\mu y(t) + f'(0) y(t-1).$$

This result is used to verify that for any two elements ϕ, ψ of $\mathcal{A}$, the difference of the solutions $x^{\phi} - x^{\psi}$ has at most one sign change in all intervals $[t - r(x^{\phi}(t)), t]$ and $[t - r(x^{\psi}(t)), t]$, $t \in \mathbb{R}$. This fact is important in the proof of the injectivity of the map

$$\Pi : \mathcal{A} \ni \phi \mapsto \begin{pmatrix} \phi(0) \\ \phi(-r(\phi(0))) \end{pmatrix} \in \mathbb{R}^2.$$

The paper [22] considers Equation (8.1.1) in the positive feedback case, i.e., $f' > 0$, and proves certain results which are analogous to the constant delay case in [135].

We remark that in the constant delay case, it is also known that the attractor of the slowly oscillating solutions is a C^1-smooth submanifold of the phase space [217]. Another remarkable result is that the domain of attraction is an open dense subset of the phase space [162]. Whether these remain true for the state-dependent delay case are open problems.

8.2. **Limiting profiles for a singular perturbation problem.** In their series of papers [154, 155, 158] Mallet-Paret and Nussbaum determine the asymptotic shape, or *limiting profile*, of periodic solutions to equations of the form

$$\epsilon\, x'(t) = f(x(t), x(t - r(x(t)))), \tag{8.2.1}$$

for $\epsilon \to 0$. A limiting profile is a subset Ω of the plane $\mathbb{R}^2$ which arises as limit of a sequence of solutions

$$x^k = \{(t, x^k(t)) \in \mathbb{R}^2 : t \in \mathbb{R}\},\ k \in \mathbb{N},$$

to Equation (8.2.1) with $\epsilon = \epsilon_k$, in case $\lim_{k\to\infty} \epsilon_k = 0$. Convergence of a sequence of closed subsets of the plane means that intersections with given compact subsets converge in the Hausdorff metric.

The standing hypotheses in [158] are the following. f is a Lipschitz continuous real-valued map defined on a square $I \times I$, $I = [-D, C]$ with $C > 0$ and $D > 0$, and $r : I \to [0, \infty)$ is Lipschitz continuous with

$$r(0) = 1 \text{ and } r(\xi) > 0 \text{ for } -D < \xi < C.$$

The zeroset $f^{-1}(0) \subset I \times I$ is a strictly decreasing continuous function $g : I \to I$ with $g(0) = 0$, and f is positive below its zeroset and negative above it. Moreover,

$$|g^2(\xi)| < |\xi| \text{ on } (-D, C) \setminus \{0\}.$$

In case $r(C) > 0$ it is assumed that $g(C) = -D$ while in case $r(-D) > 0$, $g(-D) = C$. Finally, f is differentiable at $(0, 0)$ with $D_2 f(0, 0)1 < D_1 f(0, 0)1$.

Notice that the distribution of the signs of f in $I \times I$ generalizes the negative feedback inequality for functions of a single variable.

The periodic solutions considered are slowly oscillating in the sense that their zeros are spaced at distances larger than $r(0) = 1$, which is the delay at equilibrium. Their minimal periods are given by 3 consecutive zeros, and they are sine-like in the sense that the period interval between 3 successive zeros consists of 3 adjacent subintervals on each of which the periodic solution is monotone. Existence of sine-like slowly oscillating periodic solutions for sufficiently small $\epsilon > 0$ is proved in [155].

Limiting profiles exist, due to a result from [155] that for every sequence of parameters $\epsilon_k > 0$ with $\lim_{k\to\infty} \epsilon_k = 0$ and for every sequence of sine-like slowly oscillating periodic solutions $x^k : \mathbb{R} \to I$ of Equation (8.2.1) with $\epsilon = \epsilon_k$ there is a subsequence (k_j) for which the graphs x^{k_j} converge.

It may happen that the limiting profile is simply the abscissa $\mathbb{R} \times \{0\}$. Theorem 5.1 in [155] provides sufficient conditions on r which exclude this case, like for example $r'(0) \neq 0$.

The first step towards the description of nontrivial limiting profiles is an appropriate interpretation of the formal limit of Equation (8.2.1) for $\epsilon \to 0$, which reads

$$0 = f(x(t), x(t - r(x(t)))) \tag{8.2.2}$$

and can be considered as a difference equation in implicit form for functions on the real line. But this is too narrow, as limiting profiles may contain vertical line segments. Notice that for any point $(t, x(t))$ on a solution $x : \mathbb{R} \to I$ of Equation (8.2.2) there is another point $(s, x(s))$ on x with

$$s = t - r(x(t)) \leq t \text{ and } x(s) = g(x(t))$$

since Equation (8.2.2) is solved for the second argument by g. This suggests to consider the *backdating map* $\Phi : \mathbb{R} \times I \to \mathbb{R} \times I$ given by

$$\Phi(\tau, \xi) = (\tau - r(\xi), g(\xi))$$

and its trajectories (τ_n, ξ_n), which satisfy the system

$$\begin{aligned} 0 &= f(\xi_n, \xi_{n-1}) \\ \tau_{n-1} &= \tau_n - r(\xi_n). \end{aligned}$$

Properties of the backdating map are the key to the description of limiting profiles.

Theorem 1.3 in [158] establishes that the minimal periods $p^k > 2$ of sine-like slowly oscillating periodic solutions $x^k : \mathbb{R} \to I$, $k \in \mathbb{N}$, of Equation (8.2.1) with $\epsilon = \epsilon_k$, $\lim_{k\to\infty} \epsilon_k = 0$, are bounded.

Each nontrivial limiting profile $\Omega \subset \mathbb{R} \times I$ is periodic, i. e.,

$$\Omega = \Omega + (p, 0)$$

with the (existing) limit $p \geq 2$ of the minimal periods of the approximating sequence of periodic solutions, and the intersection of Ω with a suitable vertical strip of width p can be written as the union of a horizontal line segment below the abscissa, an ascending part, a horizontal line segment above the abscissa, and a descending part. The horizontal parts may be singletons, and the ascending and descending parts may contain both horizontal and vertical line segments.

The first main result, Theorem A in [158], describes a nontrivial limiting profile Ω in the following way, using the (existing) limits $\mu > 0$ of the maxima and $-\nu < 0$ of the minima of the approximating periodic solutions: There is a sequence of continuous functions $\psi_n : [-\nu, \mu] \setminus \{0\} \to \mathbb{R}$, $n \in \mathbb{Z}$, with right and left limits at 0, so that for each integer n the function $(-1)^n \psi_n$ is increasing, $\psi_n \leq \psi_{n+1}$, $\psi_{n+2} = \psi_n + p$, and

$$\Omega = \left(\bigcup_n \psi_n^* \cup (A_n \times \{0\}) \right) \cup \left(\bigcup_n B_n \times \{\lambda_n\} \right)$$

where

$$\psi_n^* = \{(\psi_n(\xi), \xi) : 0 \neq \xi \in [-\nu, \mu]\},$$

$$\begin{aligned} A_n &= [\psi_n(0-), \psi_n(0+)] \text{ if } n \text{ is even} \\ A_n &= [\psi_n(0+), \psi_n(0-)] \text{ if } n \text{ is odd,} \end{aligned}$$

and

$$B_n = [\psi_n(\lambda_n), \psi_{n+1}(\lambda_n)]$$

with

$$\begin{aligned} \lambda_n &= \mu \text{ if } n \text{ is even,} \\ \lambda_n &= -\nu \text{ if } n \text{ is odd.} \end{aligned}$$

Moreover, there exist $\delta_0 > 0$ and $\delta_1 > 0$ so that for $0 \neq \xi \in [-\nu, \delta_0]$,

$$\psi_{2m}(\xi) = \max_{-\nu \leq s \leq \xi} (r(s) + \psi_{2m-1}(g(s))) \tag{8.2.3}$$

while for $0 \neq \xi \in [-\delta_1, \mu]$,

$$\psi_{2m+1}(\xi) = \max_{\xi \leq s \leq \mu} (r(s) + \psi_{2m}(g(s))). \tag{8.2.4}$$

The nonlocal *max-plus operators* given by the right hand sides of the equations (8.2.3-4) bear analogies with linear Fredholm integral operators. To see this, replace addition in (8.2.3-4) by multiplication and maximization by integration. In [156, 157] Mallet-Paret and Nussbaum study max-plus operators and associated eigenvalue problems; the theory is applied in [158].

Theorem B in [158] deals with monotone delay functions r and provides more detailed information about limiting profiles, in terms of f, g, r and $h = r + r \circ g$. Here the functions ψ_n for n odd are solutions to an eigenvalue problem

$$p + \psi_n(\xi) = \max_{\xi \leq s \leq \mu} (h(s) + \psi_n(g^2(s)))$$

for a max-plus operator, with the period p as an *additive eigenvalue.* The functions ψ_n for n even are computed from ψ_n for n odd.

Theorem C in [158] establishes uniqueness of limiting profiles, under further conditions on the data f and r.

A simple-looking example for which there is a unique limiting profile is the equation

$$\epsilon\, x'(t) = -x(t) - k\, x(t-1-c\,x(t))$$

with $k>1$ and $c>0$. In this case, $p=1+k$ and

$$\begin{aligned} \Omega \cap ((-1,k)\times\mathbb{R}) &= \{(\tau, \frac{1}{c}\tau) : -1<\tau<k\} \\ \Omega \cap (\{k\}\times\mathbb{R}) &= \{(k,\xi) : -\frac{1}{c} \le \xi \le \frac{k}{c}\}. \end{aligned}$$

So Ω is uniquely determined and looks like sawteeth.

8.3. **Small delay.** In [182] Ouifki and Hbid obtain existence of periodic solutions for a system of the form

$$x'(t) = g(x(t - r(x_t))) \tag{8.3.1}$$

with a map $g:\mathbb{R}^2\to\mathbb{R}^2$ which satisfies $g(0)=0$, is smooth of class C^4, and has Jacobian $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ at $0\in\mathbb{R}^2$. Also the delay functional $r: C\to[0,h]\subset\mathbb{R}$, $C=C([-h,0];\mathbb{R}^2)$, is assumed to be smooth of class C^4, and several smallness conditions are imposed. The approach in [182] is based on a decomposition of the functional $f: C\to\mathbb{R}^2$ corresponding to the right hand side of Equation (8.3.1) into a smooth map $f_r: C\to\mathbb{R}^2$ and a remainder term, both depending on r. The decomposition holds for arguments in a closed subset E_1 of the space $C^{0,1}=C^{0,1}([-h,0];\mathbb{R}^2)$. The set E_1 is contained in the analogue $E\subset C^1=C^1([-h,0];\mathbb{R}^2)\subset C^{0,1}$ of the solution manifold X_f from Section 3; in [147] it was shown that a nonlinear semigroup on $C^{0,1}$ generated by equations like (8.3.1) becomes strongly continuous if restricted to E. Smallness assumptions on r yield that the set E_1 is positively invariant. Each truncated equation

$$y'(t) = f_r(y_t) \tag{8.3.2}$$

defines a semiflow on the space C, as f_r is sufficiently smooth. The delay $\mu=r(0)$ at equilibrium is then considered as a parameter; Equation (8.3.2) is rewritten as

$$y'(t) = f_{\tilde r}(\mu, z_t), \tag{8.3.3}$$

with $\tilde r(\phi) = r(\phi) - r(0)$. Under assumptions which guarantee certain stability properties of the stationary point 0 of Equation (8.3.3) with $\mu=0$ and $\tilde r$ small, a combination of center manifold theory with a Hopf bifurcation theorem yields attracting periodic orbits $o(\mu,\tilde r)$ of the truncated equation (8.3.2) for small $\mu>0$ and small $\tilde r$. Upon that a return map is constructed following solutions of the original equation (8.3.1) which start from initial data in E_1 close to a chosen point on $o(\mu,\tilde r)$. This requires further smallness properties of $\tilde r$; closeness refers to the topology of $C^{0,1}$. With respect to this topology the return map is continuous and compact, Schauder's theorem is applied, and resulting fixed points define periodic solutions of Equation (8.3.1).

8.4. **Generic convergence.** In [21] Bartha considers the scalar equation

$$x'(t) = -\mu x(t) + f(x(t - r(x(t)))) \tag{8.4.1}$$

assuming that $\mu > 0$, $f \in C^1(\mathbb{R}, \mathbb{R})$, $f(0) = 0$, $f'(u) > 0$ for all $u \in \mathbb{R}$, $r \in C^1(\mathbb{R}, \mathbb{R})$, and there is $A > 0$ with

$$|f(u)| < \mu |u| \qquad \text{for all } u \in \mathbb{R} \setminus (-A, A).$$

In addition, it is also required that

$$r(u) > 0 \qquad \text{for all } u \in [-A, A].$$

Setting $R = \max_{u \in [-A,A]} |r(u)|$, the metric space X is defined as the space of Lipschitz continuous functions $\phi : [-R, 0] \to [-A, A]$ equipped with the metric

$$d(\phi, \psi) = \max_{s \in [-R,0]} |\phi(s) - \psi(s)|.$$

Then, for every $\phi \in X$, Equation (8.4.1) has a unique solution $x^\phi : [-R, \infty) \to [-A, A]$ with $x^\phi|_{[-R,0]} = \phi$, and the relations

$$F(t, \phi) = x_t^\phi,\ t \geq 0,\ x_t^\phi(s) = x^\phi(t + s),\ -R \leq s \leq 0,$$

define a continuous semiflow on X.

Using the standard ordering

$$\phi \leq \psi \quad \text{iff} \quad \phi(s) \leq \psi(s),\ -R \leq s \leq 0,$$

it is relatively straightforward to show that the semiflow F is monotone, i.e., $F(t, \phi) \leq F(t, \psi)$ whenever $\phi \in X$, $\psi \in X$, $\phi \leq \psi$ and $t \geq 0$. However, the strongly order preserving property (SOP), which is a crucial hypothesis in the generic convergence theorem of Smith and Thieme in [198], does not hold in general for F. Recall that SOP of F means the monotonicity of F, and that in case $\phi \in X$, $\psi \in X$, $\phi \leq \psi$, $\phi \neq \psi$ there exist $t_0 > 0$ and open subsets U, V of X with $\phi \in U$ and $\psi \in V$ such that $F(t_0, U) \leq F(t_0, V)$. Here, for subsets S, T of X we write $S \leq T$ if $\phi \leq \psi$ holds for all $\phi \in S$ and $\psi \in T$. The main reason of the failure of the SOP property for F is that for different elements ϕ, ψ of X with $\phi \leq \psi$, $x^\phi(t) = x^\psi(t)$ may happen for all $t \geq 0$.

In [21] the SOP property is replaced by the weaker mildly order preserving property (MOP). Introduce

$$\phi <_F \psi$$

for elements ϕ, ψ of X if $\phi \leq \psi$, $\phi \neq \psi$, and $F(t, \phi) \neq F(t, \psi)$ for all $t \geq 0$. Then F is said to be MOP if it is monotone, and for every ϕ, ψ in X with $\phi <_F \psi$, there exist $t_0 > 0$ and open subsets U, V of X with $\phi \in U$ and $\psi \in V$ such that $F(t_0, U) \leq F(t_0, V)$.

[21] proves that F has the MOP property. An important step toward this result is that for two globally defined solutions $x : \mathbb{R} \to [-A, A]$ and $y : \mathbb{R} \to [-A, A]$ with $x_0 = y_0$, it is true that

$$x(t) = y(t) \qquad \text{for all } t \in \mathbb{R}.$$

The abstract generic convergence result of Smith and Thieme from [198] is modified in [21] so that SOP is replaced by MOP. Then the main result of [21] is that there is an open dense subset Y of X such that

$$\lim_{t \to \infty} x^\phi(t) \quad \text{exists}$$

for all $\phi \in Y$.

8.5. **Stability and oscillation.** Several results in the literature which deal with limiting behaviour of solutions to nonautonomous differential equations with non-constant delay are also valid for equations with state-dependent delays, see, e.g., [229], [130]. Most of these papers concentrate on the behaviour of given solutions, not on questions of existence, uniqueness, and continuous dependence.

Here we list a few papers where the presence of the state-dependent delay is emphasized since it causes new technical difficulties.

Kuang [137] considers the scalar nonautonomous state-dependent delay differential equation

$$x'(t) = -g(t, x(t)) - e^{-\eta\tau(x_t)} f(t, x(t - \tau(x_t))).$$

Sharp conditions for the boundedness of solutions, global and uniform stability of the trivial solution are presented.

Cooke and Huang [48] study the scalar equation

$$x'(t) = x(t) \left(a - bx(t) - \sum_{i=1}^{L} b_i x(t - r_i) - cx(t - \tau(x_t)) \right),$$

where a, b_i, c are positive constants, τ is a functional of the history of $x(\cdot)$ over all times before t. They obtain results on convergence of positive solutions, periodic and oscillatory behaviour which extend work of G. Seifert for the constant delay case.

Győri and Hartung [95] consider the linear delay differential systems

$$x'(t) = A_i(t)x(t - \sigma_i(t)) \tag{8.5.1}$$

with continuous coefficient functions $A_i : [0, \infty) \to \mathbb{R}^{n\times n}$ and continuous delay functions $\sigma_i : [0, \infty) \to \mathbb{R}$, $i \in \{1, 2\}$, such that, for some $r > 0$, $0 \le \sigma_i(t) \le t + r$, $t \ge 0$, and $\liminf_{t\to\infty}[t - \sigma_i(t)] > 0$. Assuming that the zero solution of Equation (8.5.1) with $i = 1$ is exponentially stable, explicit neighbourhoods of A_1 and σ_1 are constructed so that if A_2 and σ_2 belong to the corresponding neighbourhoods of A_1 and σ_1, then the zero solution of Equation (8.5.1) with $i = 2$ is also exponentially stable. As an application, among others, sufficient conditions are given to guarantee the exponential stability of the zero solution of the scalar equation

$$x'(t) = a(t)x(t - \tau(t, x_t))$$

with delay functional τ defined by the threshold relation

$$\int_{t-\tau(t,x_t)}^{t} f(t, s, x_t)\, ds = m$$

provided that the zero solution of

$$x'(t) = a(t)x(t - \tau(t, 0))$$

is exponentially stable.

Cao, Fan and Gard [38] study the two-stage population model of Aiello, Freedman and Wu [2] with density-dependent delay. They show that no Hopf bifurcation can occur in the sense that the characteristic equation, associated with linearization at any strictly positive equilibrium, never has imaginary roots. Instability can arise together with the creation of multiple equilibria. The attractivity regions of the equilibrium points are also estimated.

Bélair [24] considers an age-structured model, and reduces it to a system of delay differential equations with state-dependent delay. Assuming that a center manifold reduction is valid, a supercritical Hopf bifurcation is established.

Rai and Robertson [188], [189] study stage-structured population models with delay where the delay is a function of the total population density, and they prove positivity, boundedness and stability of the solutions.

Bartha [20] addresses the stability and convergence of solutions for a class of neutral functional differential equations with state-dependent delay. The equation is transformed into a retarded differential equation with infinite delay. The state-dependent delay causes that the transformation depends on each particular solution. For the retarded equation with infinite delay a result of Krisztin [130] can be applied to get sharp sufficient conditions for the stability of the zero solution. The second part of [20] contains attractivity results.

Pinto [186] gives conditions assuring asymptotic expansions of the form

$$y(t) = \exp\left(\int_{t_0}^{t} a(s)\,ds\right)\left[\xi + O\left(\int_{t}^{\infty} \lambda(s)\,ds\right)\right] \qquad \text{as } t \to \infty$$

for the solutions of the scalar state-dependent differential equation

$$y'(t) = a(t)y(t - r(t, y(t)))$$

with certain $\xi \in \mathbb{R}$, $\lambda \in L_1[0, \infty)$ constructed by means of the coefficient function a and the delay function r. These results are extended to systems in [84].

Asymptotic solution behaviour for various classes of autonomous equations with state-dependent delays is investigated also in [46], [47], [106], [54].

Gatica and Rivero [87] obtain sufficient conditions for the oscillation of all nontrivial solutions of a scalar equation with a state-dependent delay given by a threshold condition.

Domoshnitsky, Drakhlin and Litsyn [57] study the equation

$$x'(t) + \sum_{i=1}^{m} A_i(t)x(t - (H_i x)(t)) = f(t)$$

in $\mathbb{R}^n$ with measurable and essentially bounded functions A_i and f, and measurable delay functional H_i. Sufficient conditions are obtained for the boundedness, oscillation and nonoscillation of the solutions by using an associated linear equation.

Additional stability and oscillation results can be found in [37], [47], [94], [85], [176], [177], [201], [202], [224], [207], [230], [227].

9. Numerical methods

9.1. **Preliminaries.** The study of numerical approximation for state-dependent delay equations goes back at least to the mid sixties of the last century [28, 78], and since then it is an intensively investigated research area [12, 13, 14, 17, 18, 27, 31, 36, 50, 69, 70, 71, 79, 80, 92, 127, 142, 148, 172, 173, 203, 204, 218].

In this section we concentrate on numerical methods for state-dependent delay equations (SD-DDEs) of the form

$$x'(t) = f\Big(t, x(t), x(t - \tau(t, x(t)))\Big), \qquad t \in [t_0, t_N] \tag{9.1.1}$$

with an associated initial condition

$$x(t) = \phi(t), \qquad t \in [t_0 - h, t_0]. \tag{9.1.2}$$

For simplicity we assume that $f : [t_0, t_N] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is continuous and Lipschitz continuous in its second and third variables, $\tau : [t_0, t_N] \times \mathbb{R} \to [0, h]$ is continuous, and $\phi \in C^{0,1}([t_0, t_N]; \mathbb{R})$, therefore the IVP (9.1.1)-(9.1.2) has a unique solution. The results we present can usually be easily extended to the system or multiple delays case.

We mention that there are a large number of publications dealing with other types of state-dependent differential equations including neutral SD-DDEs of the form

$$x'(t) = f\Big(t, x(t), x(t - \tau(t, x(t))), x'(t - \sigma(t, x(t)))\Big)$$

(see, e.g., [16, 27, 35, 43, 70, 121, 124, 173, 184]), the so-called "implicit" neutral SD-DDEs of the form

$$\frac{d}{dt}\Big(x(t) - g(t, x(t - \sigma(t, x(t))))\Big) = f\Big(t, x(t), x(t - \tau(t, x(t)))\Big)$$

([104, 142]), Volterra differential equations with state-dependent delays ([13, 17, 31, 36, 122, 203]), and differential-algebraic equations with state-dependent delays ([111]).

9.2. **Continuous Runge-Kutta methods for ODEs.** Before we discuss approximation of state-dependent equations, first we recall some basic notations and definitions for numerical approximation of ODEs. Consider the IVP

$$x'(t) = g(t, x(t)), \qquad t \in [t_0, t_N] \tag{9.2.1}$$

$$x(t_0) = x_0, \tag{9.2.2}$$

and suppose mesh points $\Delta = \{t_0 < t_1 < \cdots < t_N\}$ are given. *Discrete* one- or multistep methods associated to the IVP (9.2.1)-(9.2.2) produce a sequence $y_0, \ldots, y_N$ to approximate the solution x at the mesh points Δ. In this paper for simplicity we restrict our discussion to a popular class of one-step methods, the Runge-Kutta (RK) methods. A discrete RK method is defined by $y_0 = x_0$, and

$$y_{n+1} = y_n + h_n \sum_{i=1}^{s} b_i k_{n,i}, \qquad n = 0, 1, \ldots, N-1, \tag{9.2.3}$$

where the stage values $k_{n,1}, \ldots, k_{n,s}$ are determined by the system of algebraic equations

$$k_{n,i} = g\Big(t_n + h_n c_i, y_n + h_n \sum_{j=1}^{s} a_{ij} k_{n,j}\Big), \qquad i = 1, 2, \ldots, s, \tag{9.2.4}$$

and $h_n = t_{n+1} - t_n$. s is called the number of *stages*, the b_i's are the *weights*, the c_i's are the *abscissae* of the method satisfying $c_i \in [0,1]$. The coefficients a_{ij} are collected in a matrix $\mathbf{A}$, the weights and abscissae in the vectors $\mathbf{b}$ and $\mathbf{c}$, and the parameters are usually listed in the Butcher tableau $\begin{array}{c|c} \mathbf{c} & \mathbf{A} \\ \hline & \mathbf{b}^T \end{array}$. If the matrix $\mathbf{A}$ is lower triangular with zero diagonal entries then the RK method is explicit, otherwise (9.2.3) and (9.2.4) implicitly define the sequence $y_0, \ldots, y_N$.

In the mid eighties of the last century and at the beginning of the nineties the interest in the study and application of *continuous extensions* of numerical methods has been increased, since if a *dense output* is required by an ODE solver, e.g., when plotting the graph of the numerical solution, then a discrete solver is not efficient enough. Continuous methods are especially important for the approximation of delay equations where the evaluation of the approximate solution is needed in between mesh points.

One possible way to derive a continuous extension of the RK method (called CRK method) (9.2.3)-(9.2.4) is the following: Define

$$u(t_n + \theta h_n) = y_n + h_n \sum_{i=1}^{s} w_i(\theta) k_{n,i}, \qquad \theta \in [0,1], \quad n = 0, 1, \ldots, N-1, \tag{9.2.5}$$

where w_i are polynomials of degree less than or equal to δ satisfying

$$w_i(0) = 0, \qquad w_i(1) = b_i, \qquad i = 1, \ldots, s.$$

This formula is called *interpolant of the first class* of the discrete RK method, δ is the *degree of interpolant.* The Butcher tableau of a CRK method has the form $\begin{array}{c|c} \mathbf{c} & \mathbf{A} \\ \hline & \mathbf{w}^T(\theta) \end{array}$.

There are several methods to define the polynomial interpolation w_i in (9.2.5). A typical way is to use a cubic Hermite interpolation, or when a higher order interpolant is required, a so called fully Hermite interpolation [72, 112]. Another approach is to consider (9.2.4)-(9.2.5) as a discrete RK method with coefficients a_{ij}/θ, weights $w_i(\theta)/\theta$ and step size θh_n, and apply order conditions known for the discrete RK method, see, e.g., [27]. We mention that there are more general extensions of discrete RK methods than that of the form (9.2.5) (see, e.g., [27, 112]), but we do not go into details here.

Let z be the solution of the local problem

$$z'(t) = g(t, z(t)), \qquad t \in [t_n, t_{n+1}] \tag{9.2.6}$$

$$z(t_n) = z^*. \tag{9.2.7}$$

We say that the discrete RK method (9.2.3)-(9.2.4) has *order p* if $p \geq 1$ is the largest integer such that for all C^p-functions g in (9.2.1) and for all meshes Δ we have

$$|z(t_n) - y_n| = O(h_n^{p+1})$$

uniformly with respect to z^* in any bounded subset of $\mathbb{R}$ and $n = 0, \ldots, N-1$, where z is the solution of (9.2.6)-(9.2.7). Similarly, the CRK method (9.2.4)-(9.2.5) has *uniform order q* if $q \geq 1$ is the largest integer such that for all C^q-functions g in (9.2.1) and for all meshes Δ we have

$$\max\{|z(t) - u(t)| \,:\, t_n \leq t \leq t_{n+1}\} = O(h_n^{q+1})$$

uniformly with respect to z^* in any bounded subset of $\mathbb{R}$ and $n = 0, \dots, N-1$, where z is the solution of (9.2.6)-(9.2.7).

We recall the following result from [27]:

Theorem 9.2.1. *If the discrete RK method (9.2.3)-(9.2.4) has order p and if g is a C^p-function, then the method is convergent of global order p on any bounded interval $[t_0, t_N]$, i.e.,*

$$\max\{|x(t_n) - y_n| \,:\, n = 1, \dots, N\} = O(h^p),$$

where $h = \max\{h_0, \dots, h_{N-1}\}$.

Moreover, if the CRK method (9.2.4)-(9.2.5) has uniform order q, then it has uniform global order $q' = \min\{p, q+1\}$, i.e.,

$$\max\{|x(t) - u(t)| \,:\, t_0 \le t \le t_N\} = O(h^{q'}).$$

It can be checked (see [27]) that, in order to get the uniform order q, the interpolant must be of degree $\delta \ge q$, and if a discrete RK method has a continuous extension $u(t)$ of uniform order q with degree $\delta > q$, then it has a continuous extension $\tilde{u}(t)$ of uniform order q with degree $\delta = q$, as well.

As an example we present the Butcher tableau of a continuous extension of the classical four-stage discrete RK method which has uniform order 3:

$$\begin{array}{c|cccc}
0 & 0 & & & \\
\frac{1}{2} & \frac{1}{2} & 0 & & \\
\frac{1}{2} & 0 & \frac{1}{2} & 0 & \\
1 & 0 & 0 & 1 & 0 \\
\hline
 & w_1(\theta) & w_2(\theta) & w_3(\theta) & w_4(\theta)
\end{array}
\qquad \text{where} \qquad
\begin{array}{rcl}
w_1(\theta) & = & \frac{2}{3}\theta^3 - \frac{3}{2}\theta^2 + \theta \\
w_2(\theta) & = & -\frac{2}{3}\theta^3 + \theta^2 \\
w_3(\theta) & = & -\frac{2}{3}\theta^3 + \theta^2 \\
w_4(\theta) & = & \frac{2}{3}\theta^3 - \frac{1}{2}\theta^2
\end{array}$$

One-step collocation methods can be considered as CRK methods: Pick distinct abscissae $c_1, \dots, c_s \in [0, 1]$, and define

$$\begin{aligned}
\ell_i(v) &= \prod_{k=1,\ k\ne i}^{s} \frac{v - c_k}{c_i - c_k}, \qquad i = 1, \dots, s \\
a_{ij} &= \int_0^{c_i} \ell_j(v)\, dv, \qquad i, j = 1, \dots, s \\
w_i(\theta) &= \int_0^{\theta} \ell_i(v)\, dv, \qquad i = 1, \dots, s.
\end{aligned}$$

Then the corresponding CRK method (9.2.4)-(9.2.5) defines a polynomial u of degree $\le s$ satisfying the collocation equations

$$u'(t_n + h_n c_i) = g(t_n + h_n c_i, u(t_n + h_n c_i)), \qquad i = 1, \dots, s, \quad u(t_n) = y_n.$$

It is known (see, e.g., [27]) that the uniform global order of this CRK method is $q' = s$ or $s + 1$.

Nowdays an efficient differential equation solver uses a higher order continuous method, or usually a pair of higher order methods (to estimate local errors in the stepsize selection). For other examples of CRK methods we refer to [27, 50, 71, 72].

9.3. **The standard approach to approximation of SD-DDEs.** A typical approach (called "the standard approach" in [27]) to obtain a numerical approximation to the solution x of the state-dependent IVP (9.1.1)-(9.1.2) is the numerical analogue of the method of steps well-known for computing exact solutions of constant delay equations. In this subsection we describe this approach, but for simplicity, we formulate it using the class of one-step CRK methods. Clearly, it can be adopted using many other types of ODE discretization techniques.

Pick mesh points $\Delta = \{t_0 < t_1 < \cdots < t_N\}$, let $y_0 = \phi(t_0)$, and consider the sequence of local IVPs for $n = 0, 1, \ldots, N-1$:

$$
\begin{aligned}
&(9.3.1) \qquad & z_n'(t) &= f\Big(t, z_n(t), y(t - \tau(t, z_n(t)))\Big), \qquad t \in [t_n, t_{n+1}] \\
&(9.3.2) \qquad & z_n(t_n) &= y_n,
\end{aligned}
$$

where

$$
y(s) = \begin{cases} \phi(s), & s \in [t_0 - h, t_0], \\ u(s), & s \in [t_0, t_n], \\ z_n(s), & s \in [t_n, t_{n+1}]. \end{cases}
$$

Here u denotes the function $u : [t_0, t_n] \to \mathbb{R}$ whose restriction to $[t_i, t_{i+1}]$ ($i = 0, \ldots, n-1$) is the numerical solution, i.e., a CRK interpolant of the solution of the i-th IVP. Then u is already defined in the previous steps. Now we solve this IVP on $[t_n, t_{n+1}]$ using the CRK method of the form (9.2.5) corresponding to $g(t, x) = f(t, x, u(t - \tau(t, x)))$, i.e., where $k_{n,i}$'s are defined by

$$
(9.3.3) \qquad \begin{cases} t_{n,i} &= t_n + h_n c_i, \\ y_{n,i} &= y_n + h_n \displaystyle\sum_{j=1}^{s} a_{ij} k_{n,j}, \\ k_{n,i} &= f\Big(t_{n,i}, y_{n,i}, y(t_{n,i} - \tau(t_{n,i}, y_{n,i}))\Big). \end{cases}
$$

Then we extend u from $[t_0, t_n]$ to $[t_0, t_{n+1}]$ by this CRK interpolant, define $y_{n+1} = u(t_{n+1})$, and continue with the next local IVP in the same manner.

If τ is bounded below by a positive constant $\bar{\tau}$, and the stepsize is choosen so that $h_n \leq \bar{\tau}$, then (9.3.1) is an ODE, since y in (9.3.1) never takes an argument from $[t_n, t_{n+1}]$, so it is explicitly defined. Therefore if the discrete RK method is explicit, i.e., $\mathbf{A}$ is lower triangular with zeros in the diagonal, then its continuous extension (9.2.5)-(9.3.3) is also explicit.

The numerical difficulty arises in the *vanishing delay case*, i.e., when τ can be arbitrary small. Then h_n can be such that $t - \tau(t, z(t)) > t_n$ for some $t \in [t_n, t_{n+1}]$, so the evaluation of y in (9.3.3) depends also on the the unknown interpolant u on $[t_n, t_{n+1}]$. (This is called *overlapping.*) In this case the method is implicit, even if the original discrete RK method was explicit. Therefore the existence of the numerical approximation, i.e., the solvability of the algebraic equations (9.2.4)-(9.3.3) for the stage values $k_{n,1}, \ldots, k_{n,s}$ is not trivial. It can be shown ([27]) that the above problem has a positive answer: Suppose the functions f, τ and ϕ are Lipschitz continuous, then in the overlapping case there always exist a sufficiently small h_n and a suitable degree of interpolants so that the implicit relations (9.2.4)-(9.3.3) have a unique solution for the stage values $k_{n,1}, \ldots, k_{n,s}$; therefore u has a unique extension from $[t_0, t_n]$ to $[t_n, t_{n+1}]$. The proof of this result shows that this extension can be determined as a limit of a fixed point iteration, and therefore, it is common to estimate it by iteration using a predictor-corrector method. Next we show a possible algorithm to compute the n-th step of the approximation in the case when the original discrete RK method is explicit, i.e., $a_{ij} = 0$ for $i \leq j$ and $c_1 = 0$. In this algorithm we first (Step 1) predict a starting value of the stage values using the last computed approximate solution value in the overlapping case instead of interpolating values. If overlapping occurs, then in an iteration (Step 2) we correct the stage values. It can be done using a fixed number of steps (m in the algorithm), or by testing the numerical convergence in a loop. Finally (Step 3) we update the interpolant.

Step 1: Prediction – computation of initial stage values
$t_{n,1} = t_n$
$z_{n,1}^{(0)} = y_n$
$k_{n,1}^{(0)} = f(t_n, z_{n,1}^{(0)}, u(t_n - \tau(t_n, z_{n,1}^{(0)})))$
for $i = 2, \ldots, s$ do
$t_{n,i} = t_n + h_n c_i$
$$z_{n,i}^{(0)} = y_n + h_n \sum_{j=1}^{i-1} a_{ij} k_{n,j}^{(0)}$$
$d_{n,i}^{(0)} = t_{n,i} - \tau(t_{n,i}, z_{n,i}^{(0)})$
if $d_{n,i}^{(0)} \le t_n$ then
$k_{n,i}^{(0)} = f(t_{n,i}, z_{n,i}^{(0)}, u(d_{n,i}^{(0)}))$
else
$k_{n,i}^{(0)} = f(t_{n,i}, z_{n,i}^{(0)}, z_{n,i}^{(0)})$
end if
end for
Step 2: Correction by iteration is needed if $d_{n,i}^{(0)} > t_n$ was for any $i \ge 2$ in Step 1
for $r = 1, \ldots, m$ do
for $i = 1, \ldots, s$ do
$$z_{n,i}^{(r)} = y_n + h_n \sum_{j=1}^{i-1} a_{ij} k_{n,j}^{(r-1)}$$
$d_{n,i}^{(r)} = t_{n,i} - \tau(t_{n,i}, z_{n,i}^{(r-1)})$
if $d_{n,i}^{(r)} \le t_n$ then
$k_{n,i}^{(r)} = f(t_{n,i}, z_{n,i}^{(r)}, u(d_{n,i}^{(r)}))$
else
$\theta_i = \frac{d_{n,i}^{(r)} - t_n}{h_n}$
$$\hat{u}_i = y_n + h_n \sum_{j=1}^{i-1} w_j(\theta_i) k_{n,j}^{(r)} + h_n \sum_{j=i}^{s} w_j(\theta_i) k_{n,j}^{(r-1)}$$
$k_{n,i}^{(r)} = f(t_{n,i}, z_{n,i}^{(r)}, \hat{u}_i)$
end if
end for
end for
Step 3: Computation of the extension of u to $[t_n, t_{n+1}]$
$$u(t_n + \theta h_n) = y_n + h_n \sum_{i=1}^{s} w_i(\theta) k_{n,i}^{(m)}, \qquad \theta \in [0, 1]$$

Most modern differential equation solvers use non-uniform mesh size, therefore the selection of the stepsize in each integration step (the so called *primary stepsize selection*) is an important practical issue in such softwares. Concerning this topic we refer to [27, 112, 127] for more details.

The next result says that in order a method be of order p it is necessary that the solution be at least C^p on each interval $[t_n, t_{n+1}]$. We say that the function x has *discontinuity of order* p at ξ if $x^{(p-1)}$ exists and is continuous at ξ, and $x^{(p)}$ has jump discontinuity at ξ, i.e., $x^{(p)}(\xi+) \neq x^{(p)}(\xi-)$.

For the proof of the next result see [27]:

Theorem 9.3.1. *Suppose f, τ and ϕ are C^p functions. Moreover,*

(1) *the mesh $\Delta = \{t_0 < t_1 < \cdots < t_N\}$ includes all discontinuity points $\xi_1 < \xi_2 < \cdots \xi_m$ of the solution of order $\leq p$;*
(2) *the discrete RK method (9.2.1)-(9.2.3) is consistent of order p and the CRK method (9.2.4)-(9.2.5) is consistent of uniform order q.*

Then the method (9.2.4)-(9.3.3) for solving the IVP (9.1.1)-(9.1.2) is convergent of uniform global order $q' = \min\{p, q+1\}$, i.e.

$$\max\{|x(t) - u(t)| \,:\, t_0 \leq t \leq t_N\} = O(h^{q'}),$$

where $h = \max\{h_0, \ldots, h_{N-1}\}$.

We note that there are many papers which follow the basic method of steps described in this subsection, i.e., approximate the solution of an SD-DDE by that of an ODE, but defined by different discrete or continuous one- or multistep ODE solvers or with different definitions of the associated ODE [15, 50, 69, 71, 78, 93, 126, 127, 173, 184].

Rewritting a constant delay equation as an equivalent abstract Cauchy problem and using Trotter–Kato-type approximations is another popular approach especially in control applications. This technique was extended to SD-DDEs in [120].

9.4. **Tracking of derivative discontinuities.** Theorem 9.3.1 indicates that a high order method must locate all the discontinuity points of the solutions up to order p, and add them to the mesh. For constant or time-dependent delay equations this is a relatively simple task, but in the state-dependent delay case it leads to significant difficulties. Indeed, the location of the discontinuity points can not be computed *a priori* because they depend on the solution, and, on the other hand, only approximate locations can be computed.

We assume throughout this subsection that f, τ and ϕ are all C^p-functions. One can show that in case $\phi'(t_0-) \neq f(t_0, \phi(t_0), \phi(-\tau(t_0)))$ the corresponding solution x of the IVP (9.1.1)-(9.1.2) has discontinuity of order 1 at $t = t_0$, but for $t > t_0$ the solution is $C1$-smooth. If $\xi > t_0$ is such that $\xi - \tau(\xi, x(\xi)) = t_0$, it is easy to check that $x''(\xi-) \neq x''(\xi+)$, so x has a discontinuity of order 2 at $t = \xi$. Similarly, if we set $\xi_0 = t_0$ and define a (finite or infinite) sequence by the relation $\xi_{i+1} - \tau(\xi_{i+1}, x(\xi_{i+1})) = \xi_i$ for $i = 0, 1, \ldots,$ the sequence $\xi_0, \xi_1, \ldots$ consists of discontinuity points of increasing order: ξ_i has order $i + 1$. If a solution x is such that the time lag function $t \mapsto t - \tau(t, x(t))$ is strictly monotone increasing (which is satisfied in many applications), the above sequence will contain all discontinuity points of the solution.

On the other hand, if the above time lag function is not strictly monotone increasing, then the equation $\xi - \tau(\xi, x(\xi)) = \xi_i$ may have many solutions ξ. It can be checked (see [27, 79, 80, 172]) that at a solution ξ of the above equation the function x has a discontinuity of order ≥ 1, if and only if the graph of $t \mapsto t - \tau(t, x(t))$ crosses the level ξ_i, i.e., the root ξ has odd multiplicity. Therefore in this case the discontinuity points can be naturally stored in a tree: $\xi_{0,1} = t_0$ is the root of the tree, level 1 of the tree contains the solutions $\xi_{1,1}, \dots, \xi_{1,\ell_1}$ of

$$\xi - \tau(\xi, x(\xi)) = \xi_{i,j} \tag{9.4.1}$$

with odd multiplicity for $i = 0$ and $j = 1$. Then for $i = 1$ and any $j = 1, \dots, \ell_1$ the solutions $\xi_{2,1}, \dots, \xi_{2,\ell_2}$ of (9.4.1), if exist, are placed in the second level of the tree, as the descendents of the respective $\xi_{1,j}$, etc.

It was shown in [172] that there are only finitely many computationally important points (i.e., of order less or equal to p) in the discontinuity propagation tree, so they can be ordered and relabelled as an increasing sequence $t_0 = \xi_0 < \xi_1 < \xi_2 < \cdots < \xi_\ell \le t_N$. In the sequel we use this notation.

Let ξ_j be a descendent of ξ_i in the discontinuity tree, i.e.,

$$\xi_j - \tau(\xi_j, x(\xi_j)) = \xi_i. \tag{9.4.2}$$

It is easy to see that as discontinuity propagates from ξ_i to ξ_j, smoothing occurs. More precisely, we recall the next result from [172].

Theorem 9.4.1. *Suppose ξ_i and ξ_j satisfying (9.4.2) are discontinuity points of order k_i and k_j, respectively, and ξ_j has odd multiplicity m_j. Then $m_j k_i + 1 \le k_j$.*

Since the exact discontinuity points $\xi_1, \dots, \xi_\ell$ depend on the solution, suitable approximations are required. Let u denote the continuous interpolant (in the previous subsection it was a CRK interpolant) of the numerical approximation of x using certain mesh and approximate values $y_0 = \phi(t_0), y_1, \dots, y_N$. Let $\tilde{\xi}_0 = t_0$, and define the approximate discontinuity points as solutions of

$$\xi - \tau(\xi, u(\xi)) = \tilde{\xi}_i \tag{9.4.3}$$

where u is the approximate solution of the IVP satisfying $|x(\tilde{\xi}_j) - u(\tilde{\xi}_j)| = O(h^p)$, and $h = \max\{h_0, \dots, h_{N-1}\}$. The following result is cited from [79]:

Theorem 9.4.2. *Let u be a p-th order approximation of x at $\tilde{\xi}_j$, i.e., $|x(\tilde{\xi}_j) - u(\tilde{\xi}_j)| = O(h^p)$, $h = \max\{h_0, \dots, h_{N-1}\}$, m_j be the multiplicity of ξ_j in (9.4.2), and $|\xi_i - \tilde{\xi}_i| = O(h^{r_i})$. Then $|\xi_j - \tilde{\xi}_j| = O(h^{r_j})$ where $r_j = \min\{p, r_i\}/m_j$.*

Feldstein and Neves [79] suggested the following *secondary stepsize control* to select the step size of the numerical integration method so that the approximate discontinuity points are collected to the mesh to keep the global order of the method high: Suppose at the n-th step $y_0, \dots, y_n$ are defined, and the approximate discontinuity points found so far are $t_0 = \tilde{\xi}_0 < \tilde{\xi}_1 < \tilde{\xi}_2 < \cdots < \tilde{\xi}_{\ell_n} \le t_N$.

Step 1: Predict the next approximate value of the integration method by $y_{n+1} = u(t_n + h_n)$ using the continuous method u of order p and the step size h_n selected by the primary stepsize control method of u.

Step 2: For $i = 1, \dots, \ell_n$ find the first i such that

$$(t_n - \tau(t_n, y_n) - \tilde{\xi}_i)(t_{n+1} - \tau(t_{n+1}, y_{n+1}) - \tilde{\xi}_i) < 0,$$

i.e., (9.4.3) corresponding to the right hand side $\tilde{\xi}_i$ has a solution $\tilde{\xi}$. If such i exists then proceed with Step 3, otherwise we accept h_n and y_{n+1} and finish this algorithm, i.e, go to the next iterate of computing y_{n+2}.

Step 3: Using a root-finding method (e.g., bisection) combined with the definition of u find an approximate solution $\tilde{\xi}$ with the above property (if more solutions are found, the least one is used).

Step 4: Include $\tilde{\xi}$ to the new mesh $\hat{t}_0 = t_0, \ldots, \hat{t}_n = t_n, \hat{t}_{n+1} = \tilde{\xi}, \hat{t}_{n+2} = t_{n+1}$, redefine the approximate solution values $\hat{y}_0 = y_0, \ldots, \hat{y}_n = y_n, \hat{y}_{n+1} = u(\tilde{\xi})$. Restart the numerical integration from $\hat{t}_{n+1} = \tilde{\xi}$ to $\hat{t}_{n+2}$. Let $\hat{u}$ be the new interpolant on $[\hat{t}_{n+1}, t_{n+2}]$, and define $\hat{y}_{n+2} = \hat{u}(t_{n+2})$.
Step 5: Continue with the next iterate.

See [79, 80] for more details. Concerning the global order of the above method combined with a p-th order continuous numerical approximation method u Feldstein and Neves [79] showed:

Theorem 9.4.3. *Suppose the order of discontinuity of ξ_j is k_j ($j = 1, \ldots, \ell$), $\tilde{\xi}_j$ is the corresponding approximate discontinuity point, i.e., the solution of (9.4.3), $|\xi_j - \tilde{\xi}_j| = O(h^{r_j})$, $\hat{\xi}_j$ is a numerical approximation of $\tilde{\xi}_j$ with order $|\hat{\xi}_j - \tilde{\xi}_j| = O(h^{s_j})$, and the uniform order of u is p. Then*

$$\max\{|x(t) - u(t)| \,:\, t \in [t_0, t_N]\} = O(h^s) \quad \textit{where } s = \min_{j=1,\ldots,\ell}\{p, k_j r_j, k_j s_j\}.$$

Moreover, if $s_j \geq p/k_j$ for $j = 1, \ldots, \ell$, then

$$\max\{|x(t) - u(t)| \,:\, t \in [t_0, t_N]\} = O(h^p).$$

In this subsection we assumed that the parameters of the IVP, f, τ and ϕ are all C^p-functions. If they are only piecewise C^p-functions, i.e., there are points where any of the tree parameters has smaller smoothness, then starting from such a point we can build a discontinuity propagation tree similar to what we described in this subsection. Such points are called *secondary discontinuity points*. Similarly, multiple delays can also be handled (see, e.g., [27]).

We note that the extension of the notion of the discontinuity tree from the scalar case to the system case is far from beeing obvious, and was investigated by Willé and Baker [219, 220]. They associated a so called dependency network of oriented graphs to the discontinuity points, see also [27]. For more discussions on numerical problems related to tracking discontinuity points we refer to [218, 221].

Tracking discontinuities can be computationally expensive, especially when the number of discontinuity points is large. Another typical approach to handle the loss of numerical accuracy due to the presence of derivative discontinuities is the method of *defect control* developed by Enright and Hayashi [68, 70] (see also [16, 80]). In this method the size of the defect, i.e., $\max\{|u'(t) - f(t, u(t), u(t - \tau(t, u(t))))| \,:\, t \in [t_n, t_{n+1}]\}$ is monitored and its size is controlled at each step of the integration.

9.5. **Concluding remarks.** There are several software packages available for solving SD-DDEs. Without completeness we list some of them: ARCHI (Paul [184]), DDE-STRIDEL (Butcher [35]), DDVERK (Enright and Hayashi [69]), DMRODE (Neves [171]), DKLAG6 (Corwin, Sarafyan and Thompson [50]), RADAR5 (Guglielmi and Hairer [92]), SNDDELM (Jackiewicz and Lo [124]), SYSDEL (Karouri and Vaillancourt [127]). We refer to [27, 80, 204] for further discussion and comparison of available solvers for state-dependent delay equations.

In this section we discussed some problems related to the design and analysis of numerical approximation schemes for state-dependent delay equations. Other important qualitative issues, like stability of numerical methods are not discussed here, we mention [12, 27, 142] for studies in this directions. We also mention that there are many topics beyond the scope of this survey, e.g., numerical bifurcation analysis (see [67, 148]), boundary value problems (see [18]) or parameter estimation (see [14, 102, 105, 109, 168]) for state-dependent delay equations.

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